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ON THE DUNFORD-PETTIS PROPERTY

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ABSTRACT. It is shown that the Banach spaces C_{L^1} and L^1_C and their duals have the Dunford-Pettis property. Some other more local problems are also solved.

1. Introduction. We say that a Banach space X has the Dunford-Pettis (D-P) property provided for all weakly null sequences (x_n) in X and (x_n^*) in X^*, we have that $\lim x_n^*(x_n) = 0$.

We call a subset A of X weakly conditionally compact (WCC), if any sequence in A has a weak Cauchy subsequence. It was shown by H. Rosenthal [9] that A is WCC if and only if A is bounded and does not contain a sequence which is equivalent to the usual l^1-basis. It is easily seen that the D-P property means that weakly null sequences in X are uniformly convergent on WCC subsets of X^* and vice versa.

A. Grothendieck proved that spaces of continuous functions (hence also l^∞) and Lebesgue spaces $L^1(\mu)$ are D-P spaces. More generally, $\mathcal{L}^\infty$ and $\mathcal{L}^1$-spaces (see [7] for definitions) have the D-P property.

We will show that C_{L^1} and L^1_C-spaces are also D-P, which was an open question. As will be clear from what follows, the argument here is more involving and uses a "nonlinear" technique. Our results also allow us to show that l^2 is not finitely complemented in C_{L^1}, a problem raised in [3].

We use the notation (ε_i) for the sequence of Rademacher functions on [0, 1]. Let us recall the following elementary property.

LEMMA 1. *If (x_i) is a finite sequence in a Banach space X, then* $\int \|\sum_i \varepsilon_i(\omega) x_i\|\, d\omega \geq \max_i \|x_i\|$.

The next result is a particular case of a more general theorem (see [1] and [11]) and is in fact easily derived from the Rosenthal characterization theorem [1].

LEMMA 2. *If X is a Banach space and $(x_i)_{i=1,2,\ldots}$ a WCC sequence in X, then the sequence $(x_i \otimes \varepsilon_i)_{i=1,2,\ldots}$ is WCC in L^1_X.*

The only interest of Lemma 2 here is the simplification of some arguments.

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2. The basic result. Let X be a Banach space. Denote $\bigoplus_\infty X$ the l^∞-sum of spaces X. If $\xi \in \bigoplus_\infty X$, take $|\xi| = (\|\xi_1\|, \|\xi_2\|, \ldots)$ which is an element of l^∞. For $\eta \in [\bigoplus_\infty X]^*$, we let $|\eta|$ be the positive element of $(l^\infty)^*$, given by $\langle \alpha, |\eta| \rangle = \sup\{\langle \xi, \eta \rangle; \xi \in \bigoplus_\infty X \text{ and } |\xi| \leqslant \alpha\}$ for $\alpha \in l^\infty_+$. It is indeed clear that $|\eta|$ is linear and $\| \, |\eta| \, \| = \|\eta\|$.

This section is devoted to the proof of our main result, namely

THEOREM 1. *The space $\bigoplus_\infty L^1$ has the D-P property.*

The proof requires additional lemmas.

Using the same notation as presented at the beginning of the section, we show

LEMMA 3. *If A is a WCC subset of $[\bigoplus_\infty X]^*$, then $\{|\eta|; \eta \in A\}$ is equicontinuous.*

PROOF. Suppose $\{|\eta|; \eta \in A\}$ is not equicontinuous. Then there exist $\varepsilon > 0$, a sequence (η_i) in A and a sequence (M_i) of subsets of $\mathbf{N}$, such that $|\eta_i|(M_i) > \varepsilon$ and $|\eta_i|(M_j) < 3^{-j}\varepsilon$ for $j > i$. We will obtain a contradiction on the WCC of A by showing that $(\eta_i \otimes \varepsilon_i)$ is an l^1-basis. So fix an integer k and choose scalars $a_1, \ldots, a_k$. Define $N_i = M_i \setminus \bigcup_{j=i+1}^k M_j$ for $i = 1, \ldots, k$. Clearly

$$|\eta_i|(N_i) \geqslant |\eta_i|(M_i) - \sum_{j=i+1}^{k} |\eta_i|(M_j)$$

$$> \varepsilon - \sum_{j=i}^{k} 3^{-j}\varepsilon > \frac{\varepsilon}{2}.$$

Now, applying Lemma 1

$$\int \left\| \sum_{i=1}^{k} a_i \varepsilon_i(\omega) \eta_i \right\| d\omega \geqslant \sum_{j=1}^{k} \int \left| \sum_{i=1}^{k} a_i \varepsilon_i(\omega) \eta_i \right| (N_j) \, d\omega$$

$$\geqslant \sum_{j=1}^{k} |a_j| \, |\eta_j|(N_j) \geqslant \frac{\varepsilon}{2} \sum_{j=1}^{k} |a_j|,$$

proving the lemma.

The Banach space X will be $L^1(\mu)$, where μ is some probability measure.

LEMMA 4. *Let A be a WCC subset of $\bigoplus_\infty L^1$, ν a positive element of $(l^\infty)^*$ and $\varepsilon > 0$. Then there exist some ζ in $\bigoplus_\infty L^1_+$, $\|\zeta\| = 1$ and some $\delta > 0$ such that $\nu(M) < \varepsilon$ for each $\xi \in A$, where $M = \{n \in \mathbf{N}; \int_S \zeta_n \, d\mu < \delta$ and $\int_S \xi_n \, d\mu > \varepsilon$ for some $S\}$.*

PROOF. The requirement $\|\zeta\| = 1$ is of course unimportant. Assume the claim untrue. Proceeding by induction, it is possible to obtain a sequence (ξ^i) in A such that $\nu(M_i) \geqslant \varepsilon$ for each i, where

$$M_i = \left\{ n \in \mathbf{N}; \int_S \zeta_n^i \, d\mu < 3^{-i}\varepsilon \text{ and } \int_S |\xi_n^i| \, d\mu > \varepsilon \text{ for some } S \right\}$$

and $\zeta^i \in \bigoplus_\infty L^1$ is given by $\zeta_n^i = \max(|\xi_n^j|; 1 \leqslant j \leqslant i-1)$. The construction is again completely straightforward, so we omit details.

There is an infinite subset I of $\mathbf{N}$ such that $(M_i)_{i\in I}$ has the finite intersection property. We will show that $(\xi^i \otimes \varepsilon_i)_{i\in I}$ is equivalent to the l^1-basis, which will provide the required contradiction.

Fix $i_1 < i_2 < \cdots < i_r$ in I and scalars $a_1, a_2, \ldots, a_r$. Take $n \in \bigcap_{s=1}^r M_{i_s}$. For each $s = 1, \ldots, r$ there exist a set S_s satisfying $\int_{S_s} |\xi_n^{i_s}|\, d\mu > \varepsilon$ and $\int_{S_s} |\xi_n^{i}|\, d\mu < 3^{-s}\varepsilon$ for $i < i_s$. So if $T_s = S_s \setminus \bigcup_{t=s+1}^r S_t$, we obtain

$$\int_{T_s} |\xi_n^{i_s}|\, d\mu > \int_{S_s} |\xi_n^{i_s}|\, d\mu - \sum_{t=s+1}^{r} \int_{S_t} |\xi_n^{i_s}|\, d\mu$$
$$> \varepsilon - \sum_{t=s+1}^{r} 3^{-t}\varepsilon > \frac{\varepsilon}{2}.$$

Finally, again by Lemma 1

$$\int \Big\| \sum_s a_s \varepsilon_s(\omega) \xi^{i_s} \Big\|\, d\omega > \int \Big\| \sum_s a_s \varepsilon_s(\omega) \xi_n^{i_s} \Big\|\, d\omega$$
$$> \sum_{t=1}^{r} \int \int_{T_t} \Big| \sum_s a_s \varepsilon_s(\omega) \xi_n^{i_s} \Big|\, d\mu\, d\omega$$
$$> \sum_{t=1}^{r} |a_t| \int_{T_t} |\xi_n^{i_t}|\, d\mu > \frac{\varepsilon}{2} \sum_{t=1}^{r} |a_t|.$$

This completes the proof.

We assume (ξ^i) a weakly null sequence in $\bigoplus_\infty L^1$, (η_i) a weakly null sequence in $(\bigoplus_\infty L^1)^*$ such that $\|\xi^i\| = \|\eta_i\| = 1$ and $\langle \xi^i, \eta_i \rangle > \rho$ $(\rho > 0)$ and will work towards a contradiction.

If $\mathfrak{S}$ is a sub-σ-algebra, $\mu' << \mu$ and $f \in L^1(\mu')$, then $E[f, \mathfrak{S}, \mu']$ will denote the conditional expectation of f with respect to $\mathfrak{S}$ in the space $L^1(\mu')$.

LEMMA 5. *There exist θ in $\bigoplus_\infty L^1_+$, a sequence (f^j) in $\bigoplus_\infty L^\infty$ and a sequence (λ_j) in $(\bigoplus_\infty L^1)^*$ such that the following conditions are fulfilled.*

(1) $\|f^j\| = \sup_n \|f_n^j\|_\infty \leqslant 1$.

(2) *For each n, the sequence $(f_n^j)_j$ is a martingale difference sequence in the space $L^1(\mu_n)$, where $d\mu_n = \theta_n\, d\mu$.*

(3) $\langle \Delta^j, \lambda_j \rangle > \rho/4$, *where $\Delta^j \in \bigoplus_\infty L^1$ is defined by $\Delta_n^j = f_n^j \theta_n$.*

(4) (λ_j) *is a subsequence of* (η_i).

PROOF. Let $\nu_i = |\eta_i|$ and $\nu = \sum_i 2^{-i}\nu_i$. Take $\iota = \rho/12$. By Lemma 3, there is some $0 < \varepsilon < \iota$ so that $\sup_i \nu_i(M) < \iota$ if $M \subset \mathbf{N}$ and $\nu(M) < \varepsilon$. Let then ζ and δ be as in Lemma 4, applied with $A = \{\xi^i; i\}$.

Take $\theta = \delta^{-1}\zeta$. For each n and i, let $S_n^i = [|\xi_n^i| > \theta_n]$ and $\psi_n^i = (1 - \chi_n^i)\xi_n^i$, where χ_n^i is the characteristic function of S_n^i. Let also $M_i = \{n; \|\xi_n^i - \psi_n^i\|_1 > \varepsilon\}$ for each i. Clearly

$$\int_{S_n^i} \zeta_n\, d\mu < \delta \|\xi_n^i\|_1 \leqslant \delta \quad \text{and} \quad \|\xi_n^i - \psi_n^i\|_1 = \int_{S_n^i} |\xi_n^i|\, d\mu.$$

So $\nu(M_i) < \varepsilon$, by the choice of ζ and δ.

Since $\psi_n^i \leqslant \theta_n$, we may write $\psi_n^i = g_n^i \theta_n$ for some $g_n^i \in L^\infty(\mu)$; $\|g_n^i\|_\infty \leqslant 1$. We construct the f^j and λ_j inductively.

Suppose f^j obtained such that each f_n^j is $\mathcal{E}_n^j$-measurable for some finite algebra $\mathcal{E}_n^j$, where the number $d = d_j$ of atoms of $\mathcal{E}_n^j$ does not depend on n. Denote $I_{n,e}$ $(1 \leq e \leq d)$ the atoms of $\mathcal{E}_n^j$ and consider for each $e = 1, \ldots, d$ the weakly null sequence $(\alpha^i(e))_i$ in l^∞, defined by $\alpha_n^i(e) = \int_{I_{n,e}} \xi_n^i \, d\mu$.

Let B be the direct sum of d copies of the space l^∞. We may introduce the operator Γ: $B \to \bigoplus_\infty L^1(\mu)$, defined by

$$\Gamma_n(\beta(1), \ldots, \beta(d)) = \sum_{e=1}^{d} \frac{\beta_n(e)}{\mu_n(I_{n,e})} \pi_{n,e} \theta_n,$$

where $\pi_{n,e}$ is the characteristic function of $I_{n,e}$. It is indeed clear that $\|\Gamma_n(\beta(1), \ldots, \beta(d))\|_1 \leq \Sigma_e |\beta_n(e)| \leq \Sigma_e \|\beta(e)\|$ and hence Γ is bounded.

The sequence $(\Gamma^*(\eta_i))$ is weakly null in B^*. Since l^∞ is a D-P space, B is also D-P and therefore $\langle \alpha^i(1), \ldots, \alpha^i(d), \Gamma^*(\eta_i)\rangle$ tends to 0 for $i \to \infty$.

Fix some i large enough to ensure that $|\langle \sigma^i, \eta_i \rangle| < \varepsilon$, where $\sigma^i = \Gamma(\alpha^i(1), \ldots, \alpha^i(d))$. Remark that $\|\sigma_n^i\| \leq \Sigma_e |\alpha_n^i(e)| \leq \|\xi_n^i\|$ and thus $\|\sigma^i\| \leq 1$.

A standard argument allows us to construct finite algebra's $(\mathcal{E}_n^{j+1})$ such that $\mathcal{E}_n^j \subset \mathcal{E}_n^{j+1}$, $\|g_n^i - E[g_n^i, \mathcal{E}_n^{j+1}, \mu_n]\|_\infty \leq \varepsilon/\|\mu_n\|$ and the number of atoms of $\mathcal{E}_n^{j+1}$ is bounded.

Define f^{j+1} by $2f_n^{j+1} = E[g_n^i, \mathcal{E}_n^{j+1}, \mu_n] - E[g_n^i, \mathcal{E}_n^j, \mu_n]$ if $n \in N_i = \mathbf{N} \setminus M_i$ and $f_n^{j+1} = 0$ if $n \in M_i$. Take $\lambda_{j+1} = \eta_i$. It remains to verify (3). For each n,

$$\sigma_n^i - E\left[g_n^i, \mathcal{E}_n^j, \mu_n\right]\theta_n = \sum_e \frac{\int(\xi_n^i - g_n^i\theta_n)\pi_{n,e} \, d\mu}{\mu_n(I_{n,e})} \pi_{n,e}\theta_n$$

and therefore $\|\sigma_n^i - E[g_n^i, \mathcal{E}_n^j, \mu_n]\theta_n\|_1 \leq \|\xi_n^i - \psi_n^i\|_1$. So

$$\begin{aligned}
\langle 2\Delta^{j+1}, \lambda_{j+1}\rangle &= \langle (2f_n^{j+1}\theta_n)_{n\in N_i}, \eta_i\rangle \geq \langle (g_n^i\theta_n)_{n\in N_i}, \eta_i\rangle - \langle (\sigma_n^i)_{n\in N_i}, \eta_i\rangle \\
&\quad - \sup_{n\in N_i} \left\|g_n^i\theta_n - E\left[g_n^i, \mathcal{E}_n^{j+1}, \mu_n\right]\theta_n\right\|_1 - \sup_{n\in N_i}\left\|\sigma_n^i - E\left[g_n^i, \mathcal{E}_n^j, \mu_n\right]\theta_n\right\|_1 \\
&\geq \langle (\psi_n^i)_{n\in N_i}, \eta_i\rangle - \langle (\sigma_n^i)_{n\in N_i}, \eta_i\rangle - 2\varepsilon \\
&\geq \langle \xi^i, \eta_i\rangle - \langle \sigma^i, \eta_i\rangle - \langle (\xi_n^i)_{n\in M_i}, \eta_i\rangle - \langle (\sigma_n^i)_{n\in M_i}, \eta_i\rangle - 3\varepsilon \\
&\geq \rho - (\|\xi^i\| + \|\sigma^i\|)\nu_i(M_i) - 4\varepsilon \geq \rho - 2\iota - 4\varepsilon
\end{aligned}$$

or $\langle \Delta^{j+1}, \lambda_{j+1}\rangle \geq \rho/4$. This proves the lemma.

Starting from Lemma 5, we end the proof of Theorem 1.

Proof of Theorem 1. Take $0 < \kappa < \rho/4\|\theta\|$. Since (λ_j) is weakly null, it is possible to find a finitely supported sequence (a_j) of positive scalars such that $\Sigma a_j = 1$ and $\|\Sigma_j a_j \varepsilon_j \lambda_j\| \leq \kappa$ for all signs $\varepsilon_j = \pm 1$. Consequently

$$\int \left\| \sum_j a_j \varepsilon_j(\omega) \lambda_j \right\| d\omega \leq \kappa.$$

Take now for each ω the (Riesz) product

$$R_n(\omega) = \prod_j \left(1 + \varepsilon_j(\omega) f_n^j\right)\theta_n.$$

Clearly $R_n(\omega)$ is a positive function and $\|R_n(\omega)\|_1 = \int R_n(\omega)\, d\mu = \|\theta_n\|_1$, using the fact that

$$\int f_n^{j_1} f_n^{j_2} \cdots f_n^{j_r}\, d\mu_n = 0$$

whenever $j_1 < j_2 < \cdots < j_r$.

So $R(\omega) = (R_1(\omega), R_2(\omega), \ldots)$ is a member of $\bigoplus_\infty L^1$ and $\|R(\omega)\| = \|\theta\|$. Therefore

$$\begin{aligned}\frac{\rho}{4} &> \int \Big\| \sum_j a_j \varepsilon_j(\omega)\lambda_j \Big\|\, \|R(\omega)\|\, d\omega \\ &\geqslant \int \Big\langle R(\omega), \sum_j a_j \varepsilon_j(\omega)\lambda_j \Big\rangle\, d\omega \\ &= \sum_j a_j \Big\langle \int \varepsilon_j(\omega) R(\omega)\, d\omega, \lambda_j \Big\rangle.\end{aligned}$$

But $\int \varepsilon_j(\omega) R_n(\omega)\, d\omega = f_n^j \theta_n = \Delta_n^j$ and thus $\rho/4 > \sum_j a_j \langle \Delta^j, \lambda_j \rangle$, a contradiction.

3. Consequences of the main result. The following observation will allow us to prove in certain cases the D-P property of a Banach space using the local structure of the space.

PROPOSITION 2. *Let X be a Banach space and assume $X = \overline{\bigcup_n X_n}$ where (X_n) is an increasing sequence of subspaces of X. If now $\bigoplus_\infty X_n$ is a D-P space, then X also has D-P property.*

PROOF. Assume (x_i) a weakly null sequence in X and (x_i^*) a weakly null sequence in X^* such that $\langle x_i, x_i^* \rangle$ does not tend to null. It is clear that we may assume the x_i in $\bigcup_n X_n$. Since the X_n are increasing, it is possible to find a subsequence (Y_n) of (X_n), so that $x_1, \ldots, x_n$ belongs to Y_n, for each n. We will show that $\bigoplus_\infty Y_n$ fails the D-P property. Because $\bigoplus_\infty Y_n$ is complemented in $\bigoplus_\infty X_n$, also $\bigoplus_\infty X_n$ is not D-P, which will complete the proof. Denote i_n: $Y_n \to X$ the injection and p_n: $\bigoplus_\infty Y_n \to Y_n$ the projection. Consider the sequence (ξ^i) in $\bigoplus_\infty Y_n$, where the vector ξ^i is defined by

$$\begin{aligned}\xi_n^i &= 0 \quad \text{if } n < i,\\ \xi_n^i &= x_i \quad \text{if } n \geqslant i.\end{aligned}$$

Let $\mathfrak{U}$ be a free ultrafilter on $\mathbf{N}$. Introduce the sequence (η_i) in $[\bigoplus_\infty Y_n]^*$ by taking $\eta_i = \lim_{\mathfrak{U}} p_n^* i_n^*(x_i^*)$. Thus $\langle \xi^i, \eta_i \rangle = \lim_{\mathfrak{U}} \langle \xi_n^i, x_i^* \rangle = \langle x_i, x_i^* \rangle$.

It remains to verify that (ξ^i) and (η_i) are weakly null. So fix an infinite subset M of $\mathbf{N}$ and $\delta > 0$. Because (x_i) is weakly null, there is a finitely supported sequence $(\lambda_i)_{i \in M}$ of positive scalars such that $\sum_i \lambda_i = 1$ and $\|\sum_i \lambda_i \varepsilon_i x_i\| < \delta$ for all signs $\varepsilon_i = \pm 1$. Hence

$$\begin{aligned}\Big\| \sum_i \lambda_i \xi^i \Big\| &= \sup_n \Big\| \sum_{i \leqslant n} \lambda_i x_i \Big\| \\ &= \sup_n \Big\| \frac{1}{2}\Big(\sum_{i \leqslant n} \lambda_i x_i + \sum_{i > n} \lambda_i x_i \Big) + \frac{1}{2}\Big(\sum_{i \leqslant n} \lambda_i x_i - \sum_{i > n} \lambda_i x_i \Big) \Big\|\end{aligned}$$

is also bounded by δ. Since (x_i^*) is weakly null, there is a convex combination $\sum_{i\in M}\lambda_i x_i^*$ so that $\|\sum_i \lambda_i x_i^*\| \leqslant \delta$. Consequently $\|\sum_i \lambda_i p_n^* i_n^*(x_i^*)\| \leqslant \delta$ for each n and hence also $\|\sum_i \lambda_i \eta_i\| < \delta$ as required.

REMARK. Proposition 2 has no converse. Consider for instance the space $X = \bigoplus_1 l^2(n)$, thus the l^1-sum of the $l^2(n)$-spaces. Then X has the Schur property and hence the D-P property. However, if (X_n) is an increasing sequence of subspaces of X so that $X = \overline{\bigcup_n X_n}$, then $\bigoplus_\infty X_n$ is never D-P. Indeed, since X_n contain uniformly complemented Hilbert spaces of arbitrarily large dimension, $\bigoplus_\infty l^2(n)$ is a complemented subspace of $\bigoplus_\infty X_n$ and $\bigoplus_\infty l^2(n)$ fails the D-P property by Proposition 2 (in fact, l^2 is a complemented subspace of $\bigoplus_\infty l^2(n)$).

If X and Y are two Banach spaces, then

$$d(X, Y) = \inf\{\|T\|\,\|T^{-1}\|;\ T: X \to Y \text{ is an isomorphism}\}$$

is called the Banach-Mazur distance of X and Y. We also recall the definition of an ultra-product of Banach spaces.

DEFINITION. Let I be a set, $\mathfrak{U}$ a free ultra-filter on I and $(X_i)_{i\in I}$ a family of Banach spaces. Then we let $(X_i)_{\mathfrak{U}}$ be the quotient of the space $\bigoplus_\infty X_i$ by its subspace $N_{\mathfrak{U}} = \{(x_i);\ \lim_{\mathfrak{U}}\|x_i\| = 0\}$.

For more details and general properties about ultra-products, we refer the reader to [4], [5], [7] and [13]. The following result is due to J. Stern.

PROPOSITION 3. *If X is a Banach space, then X^{**} (and consequently any even dual of X) is isometric to a 1-complemented subspace of some ultra-product $X_{\mathfrak{U}}$ of X.*

The next result about the local structure of ultraproducts is straightforward from the definition (cf. [7, p. 119]).

PROPOSITION 4. *Let X be a Banach space, m a positive integer, E a finite dimensional Banach space and $\lambda < \infty$. Assume that for any subspace U of X,* dim $U = m$, *there exists a subspace V of X such that $U \subset V$ and $d(V, E) \leqslant \lambda$. Then the same holds for any ultra-product $X_{\mathfrak{U}}$ of X.*

For any positive integer p, denote E_p the l^∞-sum of p copies of the space $l^1(p)$.

LEMMA 6. *For any integer m and $\varepsilon > 0$, there exists an integer $p = p(m, \varepsilon)$ such that if U is an m-dimensional subspace of some E_q space, then there exists a subspace V of E_q satisfying $U \subset V$ and $d(V, E_p) < 1 + \varepsilon$.*

This phenomenon is the same as for l^1 and l^∞ spaces. The proof is also completely similar (cf. [7, p. 197]).

THEOREM 5. *Let X be a Banach space and $\lambda < \infty$ with the following property.*

For any finite dimensional subspace U of X, there exists a subspace V of X such that $U \subset V$ and $d(V, E_p) \leqslant \lambda$, for some p.

Then

(1) *any ultra-product $X_{\mathfrak{U}}$ of X is D-P,*

(2) *all duals of X are D-P.*

PROOF. Using Proposition 3, the second assertion is clearly a consequence of the first. As a consequence of Lemma 6, we see that in fact the following condition is satisfied.

For any integer m, there exists an integer $p = p(m)$ such that if U is a subspace of X, $\dim U = m$, then there is a subspace V of X satisfying $U \subset V$ and $d(V, E_p) < \lambda'$ ($\lambda' > \lambda$).

Now, by Proposition 4, any ultra-product $X_{\mathfrak{U}}$ has the same property. Therefore, any separable Y of $X_{\mathfrak{U}}$ is contained in a subspace Z of $X_{\mathfrak{U}}$ of the form $Z = \overline{\bigcup_n V_n}$, where (V_n) is an increasing sequence of spaces for which $d(V_n, E_{p_n}) < \lambda'$, for some sequence (p_n) of integers.

In order to show that $X_{\mathfrak{U}}$ is D-P, it is sufficient to prove that each such space Z has D-P property. By Proposition, it is enough to show that $\bigoplus_\infty V_n$ is D-P. But $\bigoplus_\infty V_n$ is isomorphic to $\bigoplus_\infty E_{p_n}$, which is a complemented subspace of $\bigoplus_\infty l^1$ and hence of $\bigoplus_\infty L^1$. So Theorem 1 completes the proof.

Let us now consider the space C_{L^1} of continuous L^1-valued functions and the space L^1_C of Bochner integrable functions with values in C. For details about L^1_C and its dual $(L^1_C)^*$, we refer the reader to [2].

As is well known, $C(K)$-spaces are $\mathfrak{L}^\infty_{1+}$-spaces and $L^1(\mu)$-spaces are $\mathfrak{L}^1_{1+}$-spaces (cf. [7, pp. 197–199]). The next result is an analogue for L^1_C and C_{L^1}-spaces and is obtained using the same techniques.

PROPOSITION 6. *The spaces C_{L^1} and $(L^1_C)^*$ have E_p-local structure. Or more precisely, if U is a finite dimensional subspace of one of these spaces and $\varepsilon > 0$, there exists a finite dimensional subspace V satisfying $U \subset V$ and $d(V, E_p) < 1 + \varepsilon$ for some integer p.*

Thus applying Theorem 5, we find

COROLLARY 7. *The spaces C_{L^1}, L^1_C and their duals are D-P spaces.*

4. Remarks and problems.

1. In fact, Theorem 4 is equivalent to the a priori weaker statement that $\bigoplus_\infty l^1(n)$ is a D-P space. However, the proof of this result does not seem easier and we also use the "Riesz-product technique".

2. Since, by Theorem 4, the space $\bigoplus_\infty E_p$ has the D-P property, it follows that the E_p does not contain uniformly complemented Hilbert spaces of arbitrarily large dimension. This solves a problem raised in [3, p. 68].

3. It is unknown if in general the D-P property of X implies the D-P property of C_X and L^1_X. Corollary 9 gives us a positive solution to this question in case X is a C or L^1-space. In fact, we may introduce the sequence $(\mathfrak{X}_n)$ of Banach spaces, taking $\mathfrak{X}_1 = C$

$$\mathfrak{X}_{n+1} = C_{\mathfrak{X}_n} \quad \text{if } n \text{ is even,}$$

$$\mathfrak{X}_{n+1} = L^1_{\mathfrak{X}_n} \quad \text{if } n \text{ is odd.}$$

Using similar techniques, it can be shown that all these spaces (and their duals) have D-P property.

4. So far, we do not know the answer to the following question: Suppose (X_n) a sequence of finite dimensional Banach spaces such that $\bigoplus_\infty X_n$ is D-P. Does $\bigoplus_\infty X_n^*$ then have the D-P property?

ADDED IN PROOF. Recently, J. Lindenstrauss remarked that the technique explained above yields a lower bound of the order $(\log n)^{1/2}$ for the projection onto an n-dimensional Hilbert space in C_{L^1}. This shows that the result in [3] is sharp.

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