ON THE DUNFORD-PETTIS PROPERTY

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Abstract. It is shown that the Banach spaces $C_{L^1}$ and $L^1_L$ and their duals have the
Dunford-Pettis property. Some other more local problems are also solved.

1. Introduction. We say that a Banach space $X$ has the Dunford-Pettis (D-P) property provided for all weakly null sequences $(x_n)$ in $X$ and $(x_n^*)$ in $X^*$, we have
that $\lim x_n^*(x_n) = 0$.

We call a subset $A$ of $X$ weakly conditionally compact (WCC), if any sequence
in $A$ has a weak Cauchy subsequence. It was shown by H. Rosenthal [9] that $A$ is
WCC if and only if $A$ is bounded and does not contain a sequence which is
equivalent to the usual $l^1$-basis. It is easily seen that the D-P property means that
weakly null sequences in $X$ are uniformly convergent on WCC subsets of $X^*$ and
vice versa.

A. Grothendieck proved that spaces of continuous functions (hence also $L^\infty$) and
Lebesgue spaces $L^1(\mu)$ are D-P spaces. More generally, $l^\infty$ and $l^1$-spaces (see [7]
for definitions) have the D-P property.

We will show that $C_{L^1}$ and $L^1_L$-spaces are also D-P, which was an open question.
As will be clear from what follows, the argument here is more involving and uses a
"nonlinear" technique. Our results also allow us to show that $l^2$ is not finitely
complemented in $C_{L^1}$, a problem raised in [3].

We use the notation $(\varepsilon_i)$ for the sequence of Rademacher functions on $[0, 1]$. Let
us recall the following elementary property.

Lemma 1. If $(x_i)$ is a finite sequence in a Banach space $X$, then
$$\int \|\sum_i \varepsilon_i(\omega)x_i\| d\omega \geq \max_i \|x_i\|.$$ 

The next result is a particular case of a more general theorem (see [1] and [11])
and is in fact easily derived from the Rosenthal characterization theorem [1].

Lemma 2. If $X$ is a Banach space and $(x_i)_{i=1,2,\ldots}$ a WCC sequence in $X$, then the
sequence $(x_i \otimes \varepsilon_i)_{i=1,2,\ldots}$ is WCC in $L^1_X$.

The only interest of Lemma 2 here is the simplification of some arguments.

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2. The basic result. Let \( X \) be a Banach space. Denote \( \bigoplus_\infty X \) the \( l^\infty \)-sum of spaces \( X \). If \( \xi \in \bigoplus_\infty X \), take \( |\xi| = (||\xi_1||, ||\xi_2||, \ldots) \) which is an element of \( l^\infty \). For \( \eta \in [\bigoplus_\infty X]^* \), we let \( |\eta| \) be the positive element of \( (l^\infty)^* \), given by \( \langle \alpha, |\eta| \rangle = \sup\{\langle \xi, \eta \rangle; \xi \in \bigoplus_\infty X \text{ and } |\xi| \leq \alpha \} \) for \( \alpha \in l^\infty_\infty \). It is indeed clear that \( |\eta| \) is linear and \( || \eta || = || \eta || \).

This section is devoted to the proof of our main result, namely

**Theorem 1.** The space \( \bigoplus_\infty L^1 \) has the D-P property.

The proof requires additional lemmas.

Using the same notation as presented at the beginning of the section, we show

**Lemma 3.** If \( A \) is a WCC subset of \( [\bigoplus_\infty X]^* \), then \( \{|\eta|; \eta \in A \} \) is equicontinuous.

**Proof.** Suppose \( \{|\eta|; \eta \in A \} \) is not equicontinuous. Then there exist \( \varepsilon > 0 \), a sequence \( (\eta_j) \) in \( A \) and a sequence \( (M_j) \) of subsets of \( \mathbb{N} \), such that \( |\eta_j|(M_j) > \varepsilon \) and \( |\eta_j|(M_j) < 3^{-j} \varepsilon \) for \( j > i \). We will obtain a contradiction on the WCC of \( A \) by showing that \( (\eta \otimes \varepsilon_j) \) is an \( l^1 \)-basis. So fix an integer \( k \) and choose scalars \( a_1, \ldots, a_k \). Define \( N_i = M_i \setminus \bigcup_{j=i}^{k} M_j \) for \( i = 1, \ldots, k \).

\[
|\eta_j|(N_i) > |\eta_j|(M_j) - \sum_{j=i+1}^{k} |\eta_j|(M_j) \\
> \varepsilon - \sum_{j=i}^{k} 3^{-j} \varepsilon > \frac{\varepsilon}{2}.
\]

Now, applying Lemma 1

\[
\int \left\| \sum_{i=1}^{k} a_i \varepsilon_i(\omega) |\eta| \right\| \, d\omega > \sum_{j=1}^{k} \int \left\| \sum_{i=1}^{k} a_i \varepsilon_i(\omega) |\eta_i|(N_j) \right\| \, d\omega \\
> \sum_{j=1}^{k} |a_j| |\eta_j|(N_j) > \frac{\varepsilon}{2} \sum_{j=1}^{k} |a_j|,
\]

proving the lemma.

The Banach space \( X \) will be \( L^1(\mu) \), where \( \mu \) is some probability measure.

**Lemma 4.** Let \( A \) be a WCC subset of \( \bigoplus_\infty L^1 \), \( \nu \) a positive element of \( (l^\infty)^* \) and \( \varepsilon > 0 \). Then there exist some \( \zeta \) in \( \bigoplus_\infty L^1_\nu \), \( ||\zeta|| = 1 \) and some \( \delta > 0 \) such that \( \nu(M) < \varepsilon \) for each \( \xi \in A \), where \( M = \{n \in \mathbb{N}; \int_S \xi_n \, d\mu < \delta \text{ and } \int_S \xi_n \, d\mu > \varepsilon \text{ for some } S \} \).

**Proof.** The requirement \( ||\zeta|| = 1 \) is of course unimportant. Assume the claim untrue. Proceeding by induction, it is possible to obtain a sequence \( (\xi'_i) \) in \( A \) such that \( \nu(M_j) > \varepsilon \) for each \( i \), where

\[
M_i = \left\{ n \in \mathbb{N}; \int_S \xi'_i \, d\mu < 3^{-i} \varepsilon \text{ and } \int_S |\xi'| \, d\mu > \varepsilon \text{ for some } S \right\}
\]

and \( \xi'_i \in \bigoplus_\infty L^1 \) is given by \( \xi'_i = \max(|\xi'_i|; 1 < j < i - 1) \). The construction is again completely straightforward, so we omit details.
There is an infinite subset $I$ of $\mathbb{N}$ such that $(M_i)_{i \in I}$ has the finite intersection property. We will show that $(\xi_i \otimes \eta_i)_{i \in I}$ is equivalent to the $l^1$-basis, which will provide the required contradiction.

Fix $i_1 < i_2 < \cdots < i_r$ in $I$ and scalars $a_1, a_2, \ldots, a_r$. Take $n \in \cap_{i=1}^{r} M_i$. For each $s = 1, \ldots, r$ there exist a set $S_s$ satisfying $\int_{S_s} |\xi_n^i| \, d\mu > \epsilon$ and $\int_{S_s} |\xi_n^i| \, d\mu < 3^{-s}\epsilon$ for $i < i_s$. So if $T_s = S_s \setminus \cup_{i=s+1}^{r} S_i$, we obtain

$$\int_{T_s} |\xi_n^i| \, d\mu > \int_{S_s} |\xi_n^i| \, d\mu - \sum_{i=s+1}^{r} \int_{S_i} |\xi_n^i| \, d\mu > \epsilon - \sum_{i=s+1}^{r} 3^{-s}\epsilon > \frac{\epsilon}{2}.$$

Finally, again by Lemma 1

$$\int \left\| \sum_{s} a_s \epsilon_s(\omega) \xi_n^i \right\| \, d\omega > \int \left\| \sum_{s} a_s \epsilon_s(\omega) \xi_n^i \right\| \, d\omega > \sum_{s=1}^{r} \int_{T_s} \left\| \sum_{s} a_s \epsilon_s(\omega) \xi_n^i \right\| \, d\mu \, d\omega > \sum_{s=1}^{r} |a_s| \int_{T_s} |\xi_n^i| \, d\mu > \frac{\epsilon}{2} \sum_{s=1}^{r} |a_s|.$$

This completes the proof.

We assume $(\xi_i)$ a weakly null sequence in $\bigoplus_{\infty} L^1$, $(\eta_i)$ a weakly null sequence in $(\bigoplus_{\infty} L^1)^*$ such that $\|\xi\| = \|\eta\| = 1$ and $\langle \xi, \eta \rangle > \rho$ ($\rho > 0$) and will work towards a contradiction.

If $\mathcal{E}$ is a sub-$\sigma$-algebra, $\mu' < \mu$ and $f \in L^1(\mu')$, then $E[f, \mathcal{E}, \mu']$ will denote the conditional expectation of $f$ with respect to $\mathcal{E}$ in the space $L^1(\mu')$.

**Lemma 5.** There exist $\theta$ in $\bigoplus_{\infty} L^1$, a sequence $(f_n)$ in $\bigoplus_{\infty} L^\infty$ and a sequence $(\lambda_j)$ in $(\bigoplus_{\infty} L^1)^*$ such that the following conditions are fulfilled.

1. $\|f\| = \sup_n \|f_n\| < 1$.
2. For each $n$, the sequence $(f_n)$ is a martingale difference sequence in the space $L^1(\mu_n)$, where $d\mu_n = \theta_n \, d\mu$.
3. $\langle \Delta' \lambda_j \rangle > \rho/4$, where $\Delta' \in \bigoplus_{\infty} L^1$ is defined by $\Delta_n = f_n/\theta_n$.
4. $\langle \lambda_j \rangle$ is a subsequence of $(\eta_i)$.

**Proof.** Let $\nu_i = |\eta_i|$ and $\nu = \sum_i 2^{-i} \nu_i$. Take $\epsilon = \rho/12$. By Lemma 3, there is some $0 < \epsilon < \upsilon$ so that $\sup_i \nu(M) < \epsilon$ if $M \subset \mathbb{N}$ and $\nu(M) < \epsilon$. Let then $\xi$ and $\delta$ be as in Lemma 4, applied with $A = \{\xi^i_1 \}; 1)${

Take $\theta = \delta^{-1} \xi$. For each $n$ and $i$, let $S_n^i = [||\xi_n^i| > \theta_n]_a$ and $\psi_n^i = (1 - \chi_n^i)\xi_n^i$, where $\chi_n^i$ is the characteristic function of $S_n^i$. Let also $M_i = \{n; ||\xi_n^i - \psi_n^i||_1 > \epsilon\}$ for each $i$. Clearly

$$\int_{S_n^i} \xi_n \, d\mu < \delta ||\xi_n||_1 < \delta \text{ and } ||\xi_n^i - \psi_n^i||_1 = \int_{S_n^i} |\xi_n^i| \, d\mu.$$

So $\nu(M_i) < \epsilon$, by the choice of $\xi$ and $\delta$.

Since $\psi_n^i < \theta_n$, we may write $\psi_n^i = g_n^i \theta_n$ for some $g_n^i \in L^\infty(\mu)$; $\|g_n^i\|_\infty < 1$. We construct the $f_n^i$ and $\lambda_n$ inductively.
Suppose $f'$ obtained such that each $f'_j$ is $\mathcal{E}_n^j$-measurable for some finite algebra $\mathcal{E}_n^j$, where the number $d = d_j$ of atoms of $\mathcal{E}_n^j$ does not depend on $n$. Denote $I_{n,e}$ $(1 < e < d)$ the atoms of $\mathcal{E}_n^e$ and consider for each $e = 1, \ldots, d$ the weakly null sequence $(\alpha'(e))$, in $l^\infty$, defined by $\alpha'_n(e) = \int_{I_{n,e}} f'_n \, dq$.

Let $B$ be the direct sum of $d$ copies of the space $l^\infty$. We may introduce the operator $\Gamma: B \to \bigoplus_\infty L^1(\mu)$, defined by
\[
\Gamma_n(\beta(1), \ldots, \beta(d)) = \sum_{e=1}^d \beta'_n(e) \pi_{n,e} \theta_n,
\]
where $\pi_{n,e}$ is the characteristic function of $I_{n,e}$. It is indeed clear that $\|\Gamma_n(\beta(1), \ldots, \beta(d))\|_1 < \Sigma_e |\beta'_n(e)| < \Sigma_e \|\beta(e)\|$ and hence $\Gamma$ is bounded.

The sequence $(\Gamma^*(\eta))$ is weakly null in $B^*$. Since $l^\infty$ is a D-P space, $B$ is also D-P and therefore $(\alpha'(1), \ldots, \alpha'(d), \Gamma^*(\eta))$ tends to 0 for $i \to \infty$.

Fix some $i$ large enough to ensure that $|\langle \sigma^i, \eta_i \rangle| < \epsilon$, where $\sigma^i = \Gamma(\alpha'(1), \ldots, \alpha'(d))$. Remark that $\|\sigma'_n\| < \Sigma_e |\alpha'_n(e)| < \|\xi'_n\|$ and thus $\|\sigma'_n\| < 1$.

A standard argument allows us to construct finite algebra’s $(\mathcal{E}_n^{i+1})$ such that $\mathcal{E}_n^{i+1} \subset \mathcal{E}_n^i$, $\|g'_n - E[g'_n, \mathcal{E}_n^{i+1}, \mu_n]\|_\infty < \epsilon / \mu_n$ and the number of atoms of $\mathcal{E}_n^{i+1}$ is bounded.

Define $f''_n$ by $2f''_n = E[g'_n, \mathcal{E}_n^{i+1}, \mu_n] - E[g'_n, \mathcal{E}_n^i, \mu_n]$ if $n \in N_i = N \setminus M_i$ and $f''_n = 0$ if $n \in M_i$. Take $\lambda_{j+1} = \eta_i$. It remains to verify (3). For each $n$,
\[
\sigma'_n - E[g'_n, \mathcal{E}_n^i, \mu_n] \theta_n = \sum_{\epsilon} \int (\xi'_n \epsilon - g'_n \theta_n) \pi_{n,e} \, dq \pi_{n,e} \theta_n,
\]
and therefore $\|\sigma'_n - E[g'_n, \mathcal{E}_n^i, \mu_n] \theta_n\|_1 < \|\xi'_n - \psi'_n\|$. So
\[
\langle 2\Delta_{i+1}^{j+1}, \lambda_{j+1} \rangle = \langle (2f''_n \theta_n)_{n \in N_i}, \eta_i \rangle - \langle (\sigma'_n)_{n \in N_i}, \eta_i \rangle - \langle (\sigma'_n)_{n \in N_i}, \eta_i \rangle - \langle (\psi'_n)_{n \in N_i}, \eta_i \rangle - 2\epsilon
\]
\[
> \langle (\xi'_n)_{n \in M_i}, \eta_i \rangle - \langle (\xi'_n)_{n \in M_i}, \eta_i \rangle - 3\epsilon
\]
\[
> \rho - (\|\xi'_n\| + \|\sigma'_n\|) \nu(M_i) - 4\epsilon > \rho - 2\epsilon - 4\epsilon
\]
or $\langle \Delta_{i+1}^{j+1}, \lambda_{j+1} \rangle > \rho / 4$. This proves the lemma.

Starting from Lemma 5, we end the proof of Theorem 1.

PROOF OF THEOREM 1. Take $0 < \kappa < \rho / 4 \|\theta\|$. Since $(\lambda_j)$ is weakly null, it is possible to find a finitely supported sequence $(a_j)$ of positive scalars such that $\Sigma a_j = 1$ and $\|\Sigma_j a_j \epsilon_j \lambda_j\| < \kappa$ for all signs $\epsilon_j = \pm 1$. Consequently
\[
\int \left\| \sum_j a_j \epsilon_j \lambda_j \right\| \, dw < \kappa.
\]
Take now for each $\omega$ the (Riesz) product
\[
R_n(\omega) = \prod_j (1 + \epsilon_j(\omega)f'_j) \theta_n.
\]
Clearly \( R_n(\omega) \) is a positive function and \( \| R_n(\omega) \|_1 = \int R_n(\omega) \, d\mu = \| \theta_\omega \|_1 \), using the fact that
\[
\int f_{n_1}^j f_{n_2}^j \ldots f_{n_k}^j \, d\mu_n = 0
\]
whenever \( j_1 < j_2 < \cdots < j_r \).

So \( R(\omega) = (R_1(\omega), R_2(\omega), \ldots) \) is a member of \( \bigoplus_\infty L^1 \) and \( \| R(\omega) \| = \| \theta \| \). Therefore
\[
\frac{\rho}{4} > \int \left| \sum_j a_j \varepsilon_j(\omega) \lambda_j \right| \| R(\omega) \| \, d\omega
> \int \left( R(\omega), \sum_j a_j \varepsilon_j(\omega) \lambda_j \right) \, d\omega
= \sum_j a_j \left( \int \varepsilon_j(\omega) R(\omega) \, d\omega, \lambda_j \right).
\]
But \( \int \varepsilon_j(\omega) R(\omega) \, d\omega = f_j^j \theta_{n_j} = \Delta_j^j \) and thus \( \rho / 4 > \sum_j a_j \langle \Delta_j^j, \lambda_j \rangle \), a contradiction.

3. Consequences of the main result. The following observation will allow us to prove in certain cases the D-P property of a Banach space using the local structure of the space.

**Proposition 2.** Let \( X \) be a Banach space and assume \( X = \bigcup_n X_n \) where \( (X_n) \) is an increasing sequence of subspaces of \( X \). If now \( \bigoplus_\infty X_n \) is a D-P space, then \( X \) also has D-P property.

**Proof.** Assume \( (x_i) \) a weakly null sequence in \( X \) and \( (x_i^*) \) a weakly null sequence in \( X^* \) such that \( \langle x_i, x_i^* \rangle \) does not tend to null. It is clear that we may assume the \( x_i \) in \( \bigcup_n X_n \). Since the \( X_n \) are increasing, it is possible to find a subsequence \( (X^n_j) \) of \( (X_n) \), so that \( x_1, \ldots, x_n \) belongs to \( X^n_n \), for each \( n \). We will show that \( \bigoplus_\infty Y_n \) fails the D-P property. Because \( \bigoplus_\infty Y_n \) is complemented in \( \bigoplus_\infty X_n \), also \( \bigoplus_\infty X_n \) is not D-P, which will complete the proof. Denote \( i_n : Y_n \rightarrow X \) the injection and \( p_n : \bigoplus_\infty Y_n \rightarrow Y_n \) the projection. Consider the sequence \( (\xi^i_\omega) \) in \( \bigoplus_\infty Y_n \), where the vector \( \xi^i_\omega \) is defined by
\[
\xi^i_\omega = \begin{cases} 0 & \text{if } n < i, \\ x_i & \text{if } n > i. \end{cases}
\]

Let \( \mathcal{A} \) be a free ultrafilter on \( \mathbb{N} \). Introduce the sequence \( (\eta_i) \) in \( [\bigoplus_\infty Y_n]^* \) by taking \( \eta_i = \lim_{\mathcal{A}} p_n^* \iota^n_\omega(x^*_i) \). Thus \( \langle \xi^i_\omega, \eta_i \rangle = \lim_{\mathcal{A}} \langle \xi^i_\omega, x^*_i \rangle = \langle x_i, x_i^* \rangle \).

It remains to verify that \( (\xi^i_\omega) \) and \( (\eta_i) \) are weakly null. So fix an infinite subset \( M \) of \( \mathbb{N} \) and \( \delta > 0 \). Because \( (x_i) \) is weakly null, there is a finitely supported sequence \( (\lambda_i)_{i \in M} \) of positive scalars such that \( \sum_i \lambda_i = 1 \) and \( \| \sum_i \lambda_i x_i \| < \delta \) for all signs \( \varepsilon_i = \pm 1 \). Hence
\[
\left\| \sum_i \lambda_i \xi^i_\omega \right\| = \sup_n \left\| \sum_{i \leq n} \lambda_i x_i \right\|
= \sup_n \left\| \frac{1}{2} \left( \sum_{i \leq n} \lambda_i x_i + \sum_{i > n} \lambda_i x_i \right) + \frac{1}{2} \left( \sum_{i \leq n} \lambda_i x_i - \sum_{i > n} \lambda_i x_i \right) \right\|.
\]
is also bounded by $\delta$. Since $(x_n^*)$ is weakly null, there is a convex combination $\sum_{i \in M} \lambda_i x_i^*$ so that $\|\sum_i \lambda_i x_i^*\| < \delta$. Consequently $\|\sum_i \lambda_i p_n^*(x_n^*)\| < \delta$ for each $n$ and hence also $\|\sum_i \lambda_i \eta_i\| < \delta$ as required.

**Remark.** Proposition 2 has no converse. Consider for instance the space $X = \bigoplus I l^2(n)$, thus the $l^1$-sum of the $l^2(n)$-spaces. Then $X$ has the Schur property and hence the D-P property. However, if $(X_n)$ is an increasing sequence of subspaces of $X$ so that $X = \bigcup_n X_n$, then $\bigoplus_\infty X_n$ is never D-P. Indeed, since $X_n$ contain uniformly complemented Hilbert spaces of arbitrarily large dimension, $\bigoplus_\infty l^2(n)$ is a complemented subspace of $\bigoplus_\infty X_n$ and $\bigoplus_\infty l^2(n)$ fails the D-P property by Proposition 2 (in fact, $l^2$ is a complemented subspace of $\bigoplus_\infty l^2(n)$).

If $X$ and $Y$ are two Banach spaces, then

$$d(X, Y) = \inf \{ \| T \| \| T^{-1} \| ; \ T: X \to Y \text{ is an isomorphism} \}$$

is called the Banach-Mazur distance of $X$ and $Y$. We also recall the definition of an ultra-product of Banach spaces.

**Definition.** Let $I$ be a set, $\mathcal{U}$ a free ultra-filter on $I$ and $(X_i)_{i \in I}$ a family of Banach spaces. Then we let $(X, \mathcal{U})$ be the quotient of the space $\bigoplus_\infty X_i$ by its subspace $N_{\mathcal{U}} = \{ (x_i); \lim_{\mathcal{U}} \| x_i \| = 0 \}$.

For more details and general properties about ultra-products, we refer the reader to [4], [5], [7] and [13]. The following result is due to J. Stern.

**Proposition 3.** If $X$ is a Banach space, then $X^{**}$ (and consequently any even dual of $X$) is isometric to a 1-complemented subspace of some ultra-product $X_{\mathcal{U}}$ of $X$.

The next result about the local structure of ultraproducts is straightforward from the definition (cf. [7, p. 119]).

**Proposition 4.** Let $X$ be a Banach space, $m$ a positive integer, $E$ a finite dimensional Banach space and $\lambda < \infty$. Assume that for any subspace $U$ of $X$, $\dim U = m$, there exists a subspace $V$ of $X$ such that $U \subset V$ and $d(V, E) < \lambda$. Then the same holds for any ultra-product $X_{\mathcal{U}}$ of $X$.

For any positive integer $p$, denote $E_p$ the $l^\infty$-sum of $p$ copies of the space $l^1(p)$.

**Lemma 6.** For any integer $m$ and $\varepsilon > 0$, there exists an integer $p = p(m, \varepsilon)$ such that if $U$ is an $m$-dimensional subspace of some $E_q$ space, then there exists a subspace $V$ of $E_q$ satisfying $U \subset V$ and $d(V, E_p) < 1 + \varepsilon$.

This phenomenon is the same as for $l^1$ and $l^\infty$ spaces. The proof is also completely similar (cf. [7, p. 197]).

**Theorem 5.** Let $X$ be a Banach space and $\lambda < \infty$ with the following property.

For any finite dimensional subspace $U$ of $X$, there exists a subspace $V$ of $X$ such that $U \subset V$ and $d(V, E_p) < \lambda$, for some $p$.

Then

1. any ultra-product $X_{\mathcal{U}}$ of $X$ is D-P,
2. all duals of $X$ are D-P.
Proof. Using Proposition 3, the second assertion is clearly a consequence of the first. As a consequence of Lemma 6, we see that in fact the following condition is satisfied.

For any integer \( m \), there exists an integer \( p = p(m) \) such that if \( U \) is a subspace of \( X \), \( \dim U = m \), then there is a subspace \( V \) of \( X \) satisfying \( U \subset V \) and \( d(V, E_p) < \lambda' (\lambda' > \lambda) \).

Now, by Proposition 4, any ultra-product \( X_{\mathbb{N}} \) has the same property. Therefore, any separable \( Y \) of \( X_{\mathbb{N}} \) is contained in a subspace \( Z \) of \( X_{\mathbb{N}} \) of the form \( Z = \bigcup_n V_n \), where \( (V_n) \) is an increasing sequence of spaces for which \( d(V_n, E_{p_n}) < \lambda' \), for some sequence \( (p_n) \) of integers.

In order to show that \( X_{\mathbb{N}} \) is D-P, it is sufficient to prove that each such space \( Z \) has D-P property. By Proposition, it is enough to show that \( \bigoplus_{\mathbb{N}} V_n \) is D-P. But \( \bigoplus_{\mathbb{N}} V_n \) is isomorphic to \( \bigoplus_{\mathbb{N}} E_{p_n} \), which is a complemented subspace of \( \bigoplus_{\mathbb{N}} l^1 \) and hence of \( \bigoplus_{\mathbb{N}} L^1 \). So Theorem 1 completes the proof.

Let us now consider the space \( L_{\mathbb{C}}^1 \) of continuous \( L^1 \)-valued functions and the space \( \mathcal{L}_{\mathbb{C}}^1 \) of Bochner integrable functions with values in \( \mathbb{C} \). For details about \( \mathcal{L}_{\mathbb{C}}^1 \) and its dual (\( \mathcal{L}_{\mathbb{C}}^1 \))^*, we refer the reader to [2].

As is well known, \( C(K) \)-spaces are \( \ell^1 \)-spaces and \( L^1(\mu) \)-spaces are \( \ell_+^1 \)-spaces (cf. [7, pp. 197–199]). The next result is an analogue for \( \mathcal{L}_{\mathbb{C}}^1 \) and \( C_{\mathbb{C}}^1 \)-spaces and is obtained using the same techniques.

**Proposition 6.** The spaces \( C_{\mathbb{C}}^1 \) and \( (\mathcal{L}_{\mathbb{C}}^1)^* \) have \( E_p \)-local structure. Or more precisely, if \( U \) is a finite dimensional subspace of one of these spaces and \( \varepsilon > 0 \), there exists a finite dimensional subspace \( V \) satisfying \( U \subset V \) and \( d(V, E_p) < 1 + \varepsilon \) for some integer \( p \).

Thus applying Theorem 5, we find

**Corollary 7.** The spaces \( C_{\mathbb{C}}^1 \), \( \mathcal{L}_{\mathbb{C}}^1 \) and their duals are D-P spaces.

4. Remarks and problems.

1. In fact, Theorem 4 is equivalent to the a priori weaker statement that \( \bigoplus_{\mathbb{N}} l^1(n) \) is a D-P space. However, the proof of this result does not seem easier and we also use the “Riesz-product technique”.

2. Since, by Theorem 4, the space \( \bigoplus_{\mathbb{N}} E_p \) has the D-P property, it follows that the \( E_p \) does not contain uniformly complemented Hilbert spaces of arbitrarily large dimension. This solves a problem raised in [3, p. 68].

3. It is unknown if in general the D-P property of \( X \) implies the D-P property of \( C_X \) and \( L_X^1 \). Corollary 9 gives us a positive solution to this question in case \( X \) is a \( C \) or \( L^1 \)-space. In fact, we may introduce the sequence \( (\mathcal{X}_n) \) of Banach spaces, taking \( \mathcal{X}_1 = C \)

\[
\mathcal{X}_{n+1} = C_{\mathcal{X}_n} \quad \text{if } n \text{ is even,}
\]

\[
\mathcal{X}_{n+1} = L_{\mathcal{X}_n}^1 \quad \text{if } n \text{ is odd.}
\]

Using similar techniques, it can be shown that all these spaces (and their duals) have D-P property.
4. So far, we do not know the answer to the following question: Suppose \((X_n)\) a sequence of finite dimensional Banach spaces such that \(\bigoplus \nolimits _{\infty} X_n\) is D-P. Does \(\bigoplus \nolimits _{\infty} X_n^*\) then have the D-P property?

**Added in proof.** Recently, J. Lindenstrauss remarked that the technique explained above yields a lower bound of the order \((\log n)^{1/2}\) for the projection onto an \(n\)-dimensional Hilbert space in \(C_L\). This shows that the result in [3] is sharp.

**Bibliography**


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