ON THE PRODUCT OF A RIESZ SET AND A SMALL $p$ SET

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Abstract. Let $\mathbb{Z}^+$ be the semigroup consisting of all nonnegative integers. By a famous theorem of Bochner, $\mathbb{Z}^+ \times \mathbb{Z}^+$ is a Riesz set in $\mathbb{Z} \oplus \mathbb{Z}$. In this paper, we prove that the product set of a Riesz set and a small $p$ set is a small $p$ set.

1. Introduction. Let $T$ be the circle group and $\mathbb{Z}^+$ the semigroup consisting of nonnegative integers. Then, by the famous F. and M. Riesz theorem, each measure on $T$ whose Fourier-Stieltjes transform vanishes off $\mathbb{Z}^+$ is absolutely continuous with respect to the Lebesgue measure on $T$. Moreover, by a well-known theorem of Bochner, each measure on $T^2$ whose Fourier-Stieltjes transform vanishes off $\mathbb{Z}^+ \times \mathbb{Z}^+$ is absolutely continuous with respect to the Lebesgue measure on $T^2$.

In this paper, we prove that the product set of a Riesz set and a small $p$ set is a small $p$ set. We use Glicksberg's ideas [1] and the theory of disintegration.

For a LCA group $G$, $C_c(G)$, $C_0(G)$, $L^1(G)$ and $M(G)$ denote the usual spaces. For a subset $E$ of $\hat{G}$, $M_E(G)$ denotes the space consisting of all measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off $E$. We denote the Haar measure on $G$ by $m_G$.

Definition. Let $G$ be a LCA group. For a positive integer $p$, a closed subset $E$ of $\hat{G}$ is called a small $p$ set if the following is satisfied:

For each $\mu \in M_E(G)$, $\mu^p (= \mu \ast \mu \ast \cdots \ast \mu$ (p times)) belongs to $L^1(G)$. In particular, a small 1 set is called a Riesz set.

Lemma 1 [4]. Let $p$ be a positive integer. Then we have

$$t_1t_2 \cdots t_p = \sum_{i=1}^{l_p} A_i \Phi_i(t_1, t_2, \ldots, t_p)^p$$

for each $(t_1, t_2, \ldots, t_p) \in C^p$, where $A_i \in C$ (complex numbers) and $\Phi_i$ are linear forms of $t_1, t_2, \ldots, t_p$.

2. Main theorem.

Theorem 1. Let $G_1$ and $G_2$ be metrizable $\sigma$-compact LCA groups. Let $E_1$ be a small $p$ set in $\hat{G}_1$ and $E_2$ a Riesz set in $\hat{G}_2$. Then $E_1 \times E_2$ is a small $p$ set in $\hat{G}_1 \oplus \hat{G}_2$.

Proof. Let $\mu$ be a measure in $M_{E_1 \times E_2}(G_1 \oplus G_2)$. Let $\pi$ be the projection from $G_1 \oplus G_2$ onto $G_2$. Put $\eta = \pi(\{\mu\})$ (continuous image under $\pi$). Then, by disintegration theory, there exists a family $\{\lambda_\alpha\}_{\alpha \in G_2}$ in $M(G_1 \oplus G_2)$ with the following properties:

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(1) $h \mapsto \lambda_h(f)$ is a Borel measurable function of $h$ for each bounded Borel measurable function $f$ on $G_1 \oplus G_2$,

(2) $\text{supp}(\lambda_h) \subset \pi^{-1}(\{h\}) = G_1 \times \{h\}$,

(3) $\|\lambda_h\| < 1$,

(4) $\mu(g) = \int_{G_2} \lambda_h(g) \, d\eta(h)$ for each bounded Borel measurable function $g$ on $G_1 \oplus G_2$.

By (2), we have $d\lambda_h(x, y) = d\nu_h(x) \times d\delta_h(y)$, where $\nu_h \in M(G_1)$ and $\delta_h$ is the Dirac measure at $h$. By the method used in [1, pp. 425-426], we have $\nu_h \in M_{E_1}(G_1)$ a.a. $h(\eta)$.

That is, there exists a Borel measurable set $K$ in $G_2$ with $\eta(G_2 \setminus K) = 0$ such that $\nu_h \in M_{E_1}(G_1)$ for $h \in K$.

Claim 1. $\eta$ belongs to $L^1(G_2)$.

For $\gamma_0 = (\gamma_1, \gamma_2) \in G_1 \oplus G_2$, we have $\pi(\gamma_0 \mu) = \mu(-\gamma_1, \gamma - \gamma_2)$. Hence $\pi(\gamma_0 \mu)$ belongs to $M_{E_1 + E_2}(G_2)$. Since $E_2$ is a Riesz set, $E_2 + E_2$ is also a Riesz set. Hence $\pi(\gamma_0 \mu)$ belongs to $L^1(G_2)$. On the other hand, there exists a sequence $\{p_n\}$ in $\text{Trig}(G_1 \oplus G_2)$ such that $\lim_{n \to \infty} \|p_n - \mu\| = 0$. Hence we have

$$\lim_{n \to \infty} \|\pi(p_n \mu) - \pi(\mu)\| = 0.$$ 

Thus Claim 1 is proved.

Claim 2. $(h_1, \ldots, h_p) \mapsto \lambda_{h_1} \times \cdots \times \lambda_{h_p}(g) = (\nu_{h_1} \times \cdots \times \nu_{h_p}) \times \delta_{(h_1+\cdots+h_p)}(g)$ is a Borel measurable function on $G_2^p$ for each bounded Borel function $g$ on $G_1 \oplus G_2$.

Let $f_j \in C_c(G_1 \oplus G_2)$ and put $f(x_1, \ldots, x_p) = f_1(x_1) \cdots f_p(x_p)$ for $(x_1, \ldots, x_p) \in (G_1 \oplus G_2)^p$. Then $(\lambda_{h_1} \times \cdots \times \lambda_{h_p})(f) = \lambda_{h_1}(f_1) \cdots \lambda_{h_p}(f_p)$ so that by (1)

\begin{equation}
(5) \quad (h_1, \ldots, h_p) \mapsto (\lambda_{h_1} \times \cdots \times \lambda_{h_p})(f) \text{ is Borel measurable.}
\end{equation}

Since $\{\sum_i f_i(x_1) \cdots f_p(x_p); f_i \in C_c(G_1 \oplus G_2)\}$ is dense in $C_0((G_1 \oplus G_2)^p)$, (5) is Borel measurable for $f \in C_0((G_1 \oplus G_2)^p)$, hence, also for bounded Borel functions $f$ on $(G_1 \oplus G_2)^p$. Let $\pi_p(x_1, \ldots, x_p) = x_1 + \cdots + x_p$ for $(x_1, \ldots, x_p) \in (G_1 \oplus G_2)^p$. Then for bounded Borel $g$ on $G_1 \oplus G_2$ we have $\lambda_{h_1} \times \cdots \times \lambda_{h_p}(g) = (\lambda_{h_1} \times \cdots \times \lambda_{h_p})(g \circ \pi_p)$ and the claim follows.

Hence, by Claim 2, we can define a measure $\xi$ in $M(G_1 \oplus G_2)$ as follows

$$\xi(f) = \int_{G_2} \int_{G_2} \cdots \int_{G_2} \left\{ (\nu_{h_1} \times \nu_{h_2} \times \cdots \times \nu_{h_p}) \times \delta_{(h_1 + h_2 + \cdots + h_p)} \right\} \times (f) \, d\eta(h_1) \, d\eta(h_2) \cdots d\eta(h_p)$$

for $f \in C_0(G_1 \oplus G_2)$.

Claim 3. $\xi = \mu^p$.

Let $(\gamma_1, \gamma_2)$ be in $G_1 \oplus G_2$. Then we have

$$\tilde{\xi}(\gamma_1, \gamma_2) = \prod_{i=1}^{p} \int_{G_2} \hat{\nu}_h(\gamma_1) \hat{\mu}(\gamma_2) \, d\eta(h)$$

$$= \{ \hat{\mu}(\gamma_1, \gamma_2) \}^p = (\mu^p)(\gamma_1, \gamma_2).$$

Thus we have $\xi = \mu^p$.

Let $E$ be a Borel measurable set in $G_1 \oplus G_2$ with $m_{G_1 \oplus G_2}(E) = 0$, where $m_{G_1 \oplus G_2}$
denotes the Haar measure on $G_1 \oplus G_2$. Then there exists a Borel measurable set $F_2$ in $G_2$ such that (i) $m_{G_2}(F_2) = 0$ and (ii) $m_{G_1}(E_y) = 0$ if $y \not\in F_2$, where $E_y = \{x \in G_1; (x, y) \in E\}$. Let $\alpha_p$ be the homomorphism from $G_2^p$ onto $G_2$ such that $\alpha_p(h_1, h_2, \ldots, h_p) = h_1 + h_2 + \cdots + h_p$. We note that $\nu_{h_1} \cdot \nu_{h_2} \cdot \cdots \cdot \nu_{h_p}$ belongs to $L^1(G_1)$ for $(h_1, h_2, \ldots, h_p) \in K^p$ by Lemma 1 and 
$$\eta^p(F_2) = (\eta \times \eta \times \cdots \times \eta)(\alpha_p^{-1}(F_2)).$$

Hence, by Claim 1 and Claim 3, we have
$$\mu^p(E) = \int_{G_2^p} \cdots \int_{G_2} \left\{ \left( \nu_{h_1} \cdot \nu_{h_2} \cdot \cdots \cdot \nu_{h_p} \right) \times \delta_{(h_1 + h_2 + \cdots + h_p)} \right\}$$
$$\times (\chi_E) \, d\eta(h_1) \, d\eta(h_2) \cdots \, d\eta(h_p)$$
$$= \int_{K^p} \left\{ \left( \nu_{h_1} \cdot \nu_{h_2} \cdot \cdots \cdot \nu_{h_p} \right) \times \delta_{(h_1, h_2, \ldots, h_p)} \right\}$$
$$= \int_{K^p \cap \alpha_p^{-1}(F_2)} \left\{ \left( \nu_{h_1} \cdot \nu_{h_2} \cdot \cdots \cdot \nu_{h_p} \right) \times \delta_{(h_1, h_2, \ldots, h_p)} \right\} + \int_{K^p \setminus \alpha_p^{-1}(F_2)} \left\{ \left( \nu_{h_1} \cdot \nu_{h_2} \cdot \cdots \cdot \nu_{h_p} \right) \times \delta_{(h_1, h_2, \ldots, h_p)} \right\} = 0.$$

Hence we have $\mu^p \in L^1(G_1 \oplus G_2)$. Thus $E_1 \times E_2$ is a small $p$ set in $G_1 \oplus G_2$. Q.E.D.

Dr. S. Saeki kindly pointed out the following Lemma 2 to the author.

**Lemma 2.** Suppose the product set $E_1 \times E_2$ of a small $p$ set $E_1$ in $\hat{G}_1$ and a Riesz set $E_2$ in $\hat{G}_2$ is a small $p$ set in $\hat{G}_1 \oplus \hat{G}_2$ for all metrizable LCA groups $G_1$ and $G_2$. Then the product set $E_3 \times E_4$ of a small $p$ set $E_3$ in $\hat{G}_3$ and a Riesz set $E_4$ in $\hat{G}_4$ is a small $p$ set in $G_3 \oplus G_4$ for all LCA groups $G_3$ and $G_4$.

**Proof.** Suppose there exists a measure $\mu$ in $M_{E_1 \times E_2}(G_3 \oplus G_4)$ such that $\mu^p$ does not belong to $L^1(G_3 \oplus G_4)$. Then, by Corollary 3 of [2], there exists a measure $\sigma$ in $M_{\sigma}(G_3 \oplus G_4)$ such that
$$\left( \mu * \sigma \right)^p = \mu^p * \sigma^p \not\in L^1(G_3 \oplus G_4). \quad (A)$$

Since $\sigma \in M_{\sigma}(G_3 \oplus G_4)$, there exist open $\sigma$-compact subgroups $\Gamma_i$ of $\hat{G}_i$ such that $\text{supp}(\sigma) \subset \Gamma_3 \times \Gamma_4$ (i = 3, 4). On the other hand, $\hat{G}_3$ and $\hat{G}_4$ are metrizable LCA groups. Evidently $E_3 \cap \Gamma_3$ is a small $p$ set in $\Gamma_3$ and $E_4 \cap \Gamma_4$ is a Riesz set in $\Gamma_4$. Hence, by the hypothesis of this Lemma, $(E_3 \cap \Gamma_3) \times (E_4 \cap \Gamma_4)$ is a small $p$ set in $\Gamma_3 \oplus \Gamma_4$. Therefore, since $\sigma * \mu \in M_{E_3 \cap \Gamma_3 \times E_4 \cap \Gamma_4}(G_3 \oplus G_4)$, we have $(\sigma * \mu)^p \in L^1(G_3 \oplus G_4)$.

This contradicts (A). Q.E.D.

**Lemma 3.** Let $G_3$ and $G_4$ be metrizable LCA groups. Let $E_3$ be a small $p$ set in $\hat{G}_3$ and $E_4$ a Riesz set in $\hat{G}_4$. Then $E_3 \times E_4$ is a small $p$ set in $\hat{G}_3 \oplus \hat{G}_4$.

**Proof.** Let $\mu$ be a measure in $M_{E_3 \times E_4}(G_3 \oplus G_4)$. Then there exist open metrizable $\sigma$-compact subgroups $G_1 \subset G_3$ and $G_2 \subset G_4$ such that $\text{supp}(\mu)$ is contained in $G_1 \oplus G_2$. Let $\pi_{G_2}$ be the projection from $G_1 \oplus G_2$ onto $G_2$. Put $\eta' = \pi_{G_2}(\{\mu\})$. Then,
by disintegration theory, there exists a family \( \{ \lambda'_h \}_{h \in G_2} \) in \( M(G_1 \oplus G_2) \) such that

1. \( h \mapsto \lambda'_h(f) \) is a Borel measurable function of \( h \) for each bounded Borel measurable function \( f \) on \( G_1 \oplus G_2 \),
2. \( \text{supp}(\lambda'_h) \subset G_1 \times \{ h \} \),
3. \( \| \lambda'_h \| < 1 \) and
4. \( \mu(g) = \int_{G_2} \lambda'_h(g) \, d\eta(h) \) for each bounded Borel measurable function \( g \) on \( G_1 \oplus G_2 \).

Since \( G_1 \oplus G_2 \) is \( \sigma \)-compact and metrizable, there exists a countable dense set \( \mathcal{D} = \{ f_m \} \) in \( C_0(G_1 \oplus G_2) \). For each \( g \in C_0(G_1 \oplus G_2) \), we define a function \( \tilde{g} \) in \( C_0(G_3 \oplus G_4) \) by \( \tilde{g}(x) = g(x) \) for \( x \in G_1 \oplus G_2 \) and \( \tilde{g}(x) = 0 \) for \( x \not\in G_1 \oplus G_2 \). We define measures \( \eta \in M(G_3) \) and \( \lambda_h \in M(G_3 \oplus G_4) (h \in G_4) \) as follows

\[
\eta(F) = \begin{cases} 
\lambda'_h(F' \cap G_1 \oplus G_2) & \text{if } h \in G_2, \\
0 & \text{if } h \in G_4 \setminus G_2, 
\end{cases}
\]

for a Borel measurable set \( F' \) in \( G_3 \oplus G_4 \). Then we have the following

5. \( h \mapsto \lambda'_h(f) \) is a Borel measurable function of \( h \) for each bounded Borel measurable function \( f \) on \( G_3 \oplus G_4 \),
6. \( \text{supp}(\lambda_h) \subset G_3 \times \{ h \} \subset G_3 \times \{ h \} \),
7. \( \| \lambda_h \| < 1 \),
8. \( \mu(g) = \int_{G_4} \lambda_h(g) \, d\eta(h) \) for each bounded Borel measurable function \( g \) on \( G_3 \oplus G_4 \).

From (7), we have \( d\lambda_h(x, y) = d\tilde{\rho}_h(x) \times d\tilde{\eta}_h(y) \), where \( \tilde{\rho}_h \) is a measure in \( M(G_3) \) with \( \text{supp}(\tilde{\rho}_h) \subset G_1 \). Noting that \( \text{supp}(\eta) \subset G_2 \), we may apply Lusin's theorem and regularity of \( \eta \) to obtain for each positive integer \( n \) a compact subset \( K_n \) of \( \text{supp}(\eta) \) such that

1. \( \eta(G_2 \setminus K_n) < 1/n \),
2. \( h \mapsto \lambda_h(f_m) \) is a continuous function on \( K_n \) for each \( f_m \in \mathcal{D} = \{ f; f \in \mathcal{D} \} \), and
3. for each \( x \in K_n \) and neighborhood \( V \) of \( x \), \( \eta(V \cap K_n) > 0 \).

Since \( \mathcal{D} \) is dense in \( C_0(G_1 \oplus G_2) = \{ \tilde{f}; f \in C_0(G_1 \oplus G_2) \} \) we may replace (ii) by

(ii)' \( h \mapsto \lambda_h(f) \) is continuous on \( K_n \) for each \( f \in C_0(G_1 \oplus G_2) \).

Claim 1. \( \tilde{\rho}_h \in M_{E_2}(G_3) \) for \( h \in K_n \).

Let \( f \in L^1(\tilde{G}_3) \) with \( \text{supp}(f) \subset E_2^\circ \). Then since \( \tilde{\mu}(\gamma_1, \gamma_2) = 0 \) for \( \gamma_1 \not\in E_3 \) we have

\[
0 = \int_{\tilde{G}_3} \tilde{\mu}(\gamma_1, \gamma_2) f(\gamma_1) \, d\gamma_1
= \int_{\tilde{G}_3} \int_{G_4} \int_{G_3} (-x, \gamma_1) d\tilde{\rho}_h(x)(-h, \gamma_2) \, d\tilde{\eta}(h) f(\gamma_1) \, d\gamma_1
= \int_{G_4} \int_{G_3} \tilde{f}(x) \, d\tilde{\rho}_h(x)(-h, \gamma_2) \, d\tilde{\eta}(h)
= \int_{G_4} \tilde{\rho}_h(\tilde{f})(-h, \gamma_2) \, d\tilde{\eta}(h).
\]
Hence, for each $F \in L^1(\hat{G}_4)$ and $f \in L^1(\hat{G}_2)$ with $\text{supp}(f) \subset E^c_5$ we have

$$0 = \int_{\hat{G}_4} \int_{\hat{G}_2} \nu_x(f)(-h, \gamma_2) \, d\eta(h) \, F(\gamma_2) \, d\gamma_2$$

$$= \int_{\hat{G}_4} \nu_x(f) \hat{F}(h) \, d\eta(h). \quad (11)$$

Since $L^1(\hat{G}_4)$ is dense in $C_0(\hat{G}_4)$, hence also in $L^1(\eta)$, (11) holds for all $F \in L^1(\eta)$. It follows from (6) that $h \mapsto \nu_x(f)$ is bounded and Borel measurable, hence in $L^\infty(\eta)$, and so by (11)

$$\nu_x(f) = 0, \quad \eta\text{-a.e.} \quad (12)$$

Let $\beta \in C_c(\hat{G}_4)$ satisfy $\beta = 1$ on $K_n$, $\beta = 0$ off $G_2$, and set $g(x, y) = (f \ast m_{G_1})^y(x)\beta(y)$, where $G_1^\perp$ is the annihilator of $G_1$. Then $g \in C_0(G_1 \oplus G_2)$ and, since $\text{supp}(\nu_x) \subset G_1$, we have $\lambda_h(g) = \nu_x((f \ast m_{G_1}))^y\delta_h(\beta) = \nu_x(f)$ for each $h \in K_n$. Hence, $h \mapsto \nu_x(f)$ is continuous on $K_n$ by (iii), and this together with (12) and (iii) shows that $\nu_x(f) = 0$ for each $h \in K_n$. Thus, for $h \in K_n$, $0 = \nu_x(f) = \int_{\hat{G}_4} \delta_h(\nu_x(f))(h) \, d\gamma_1$.

Since $f$ is any function in $L^1(\hat{G}_3)$ with $\text{supp}(f) \subset E^c_5$ we have $\nu_x(f) = 0$ on $E^c_5$, and the claim is established. Moreover, since $\eta(G_4 \setminus \bigcup_{n=1}^\infty K_n) = 0$ we have proved that $\nu_x \in M_{\hat{G}}(G_4)$ a.a. $\eta(\gamma_1)$.

Since $E$ is a Riesz set, we may prove that $\eta \in L^1(\hat{G}_4)$ by arguing as in Theorem 1.

Claim 2. $(h_1, h_2, \ldots, h_p) \mapsto \{(\nu_{h_1} \ast \nu_{h_2} \ast \cdots \ast \nu_{h_p}) \times \delta_{(h_1+h_2+\cdots+h_p)}\}(g)$ is a Borel measurable function on $(\hat{G}_4)^p$ for each $g \in C_0(G_3 \oplus \hat{G}_4)$.

Indeed, since $\nu_x = 0$ if $h \notin G_2$, we have $\nu_{h_1} \ast \nu_{h_2} \ast \cdots \ast \nu_{h_p} \times \delta_{(h_1+h_2+\cdots+h_p)} = 0$ for $(h_1, h_2, \ldots, h_p) \notin (G_2)^p$.

On the other hand, $\lambda_h = \nu_h \times \delta_h$ may be regarded as a measure in $M(G_1 \oplus G_2)$.

Since $G_1$ and $G_2$ are $\sigma$-compact metrizable LCA groups, we may prove Claim 2 by arguing as in the proof of Claim 2, Theorem 1.

We now define a measure $\xi$ in $M(G_3 \oplus \hat{G}_4)$ as follows

$$\xi(f) = \int_{\hat{G}_4} \cdots \int_{\hat{G}_4} \{(\nu_{h_1} \ast \cdots \ast \nu_{h_p}) \times \delta_{(h_1+\cdots+h_p)}\}(f) \, d\eta(h_1) \cdots d\eta(h_p)$$

for $f \in C_0(G_3 \oplus \hat{G}_4)$.

Then we have $\xi = \mu^p$. Let $E_0$ be a Borel measurable set in $G_3 \oplus G_4$ with $m_{G_3 \oplus G_4}(E_0) = 0$. Put $E = E_0 \cap G_1 \oplus G_2$. Since $\text{supp}(\mu^p) \subset G_1 \oplus G_2$, we have $\mu^p(E_0) = \mu^p(E)$. Moreover, $m_{G_1 \oplus G_2}(E) = 0$. Thus we can prove that $\mu^p(E_0) = 0$ by using the techniques employed in Theorem 1. That is $\mu^p \in L^1(G_3 \oplus G_4)$. This completes the proof.

From Lemma 2 and Lemma 3, we obtain the following main theorem.

**Theorem 2.** Let $G_1$ and $G_2$ be LCA groups. Let $E_1$ be a small $p$ set in $\hat{G}_1$ and $E_2$ a Riesz set in $\hat{G}_2$. Then $E_1 \times E_2$ is a small $p$ set in $\hat{G}_1 \oplus \hat{G}_2$. 

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REFERENCES


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