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ON THE PRODUCT OF A RIESZ SET AND A SMALL p SET

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ABSTRACT. Let Z^+ be the semigroup consisting of all nonnegative integers. By a famous theorem of Bochner, $Z^+ \times Z^+$ is a Riesz set in $Z \oplus Z$. In this paper, we prove that the product set of a Riesz set and a small p set is a small p set.

1. Introduction. Let T be the circle group and Z^+ the semigroup consisting of nonnegative integers. Then, by the famous F. and M. Riesz theorem, each measure on T whose Fourier-Stieltjes transform vanishes off Z^+ is absolutely continuous with respect to the Lebesgue measure on T. Moreover, by a well-known theorem of Bochner, each measure on T^2 whose Fourier-Stieltjes transform vanishes off $Z^+ \times Z^+$ is absolutely continuous with respect to the Lebesgue measure on T^2.

In this paper, we prove that the product set of a Riesz set and a small p set is a small p set. We use Glicksberg's ideas [1] and the theory of disintegration.

For a LCA group G, $C_c(G)$, $C_0(G)$, $L^1(G)$ and $M(G)$ denote the usual spaces. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space consisting of all measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. We denote the Haar measure on G by m_G.

DEFINITION. Let G be a LCA group. For a positive integer p, a closed subset E of $\hat{G}$ is called a small p set if the following is satisfied:

For each $\mu \in M_E(G)$, μ^p $(= \mu * \mu * \cdots * \mu$ (p times)) belongs to $L^1(G)$. In particular, a small 1 set is called a Riesz set.

LEMMA 1 [4]. *Let p be a positive integer. Then we have*

$$t_1 t_2 \ldots t_p = \sum_{i=1}^{l_p} A_i \Phi_i(t_1, t_2, \ldots, t_p)^p \text{ for each } (t_1, t_2, \ldots, t_p) \in C^p,$$

where $A_i \in C$ (complex numbers) and Φ_i are linear forms of $t_1, t_2, \ldots, t_p$.

2. Main theorem.

THEOREM 1. *Let G_1 and G_2 be metrizable σ-compact LCA groups. Let E_1 be a small p set in $\hat{G}_1$ and E_2 a Riesz set in $\hat{G}_2$. Then $E_1 \times E_2$ is a small p set in $\widehat{G_1 \oplus G_2}$.*

PROOF. Let μ be a measure in $M_{E_1 \times E_2}(G_1 \oplus G_2)$. Let π be the projection from $G_1 \oplus G_2$ onto G_2. Put $\eta = \pi(|\mu|)$ (continuous image under π). Then, by disintegration theory, there exists a family $\{\lambda_h\}_{h \in G_2}$ in $M(G_1 \oplus G_2)$ with the following properties:

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(1) $h \mapsto \lambda_h(f)$ is a Borel measurable function of h for each bounded Borel measurable function f on $G_1 \oplus G_2$,

(2) $\operatorname{supp}(\lambda_h) \subset \pi^{-1}(\{h\}) = G_1 \times \{h\}$,

(3) $\|\lambda_h\| \leq 1$,

(4) $\mu(g) = \int_{G_2} \lambda_h(g)\, d\eta(h)$ for each bounded Borel measurable function g on $G_1 \oplus G_2$.

By (2), we have $d\lambda_h(x, y) = d\nu_h(x) \times d\delta_h(y)$, where $\nu_h \in M(G_1)$ and δ_h is the Dirac measure at h. By the method used in [1, pp. 425–426], we have $\nu_h \in M_{E_1}(G_1)$ a.a. $h(\eta)$.

That is, there exists a Borel measurable set K in G_2 with $\eta(G_2 \setminus K) = 0$ such that $\nu_h \in M_{E_1}(G_1)$ for $h \in K$.

Claim 1. η belongs to $L^1(G_2)$.

For $\gamma_0 = (\gamma_1, \gamma_2) \in \widehat{G_1 \oplus G_2}$, we have $\pi(\gamma_0 \mu)\hat{}(\gamma) = \hat{\mu}(-\gamma_1, \gamma - \gamma_2)$. Hence $\pi(\gamma_0 \mu)$ belongs to $M_{\gamma_2 + E_2}(G_2)$. Since E_2 is a Riesz set, $\gamma_2 + E_2$ is also a Riesz set. Hence $\pi(\gamma_0 \mu)$ belongs to $L^1(G_2)$. On the other hand, there exists a sequence $\{p_n\}$ in $\operatorname{Trig}(G_1 \oplus G_2)$ such that $\lim_{n\to\infty} \| p_n \mu - |\mu| \| = 0$. Hence we have

$$\lim_{n\to\infty} \|\pi(p_n \mu) - \pi(|\mu|)\| = 0.$$

Thus Claim 1 is proved.

Claim 2. $(h_1, \ldots, h_p) \mapsto \lambda_{h_1} * \cdots * \lambda_{h_p}(g) = (\nu_{h_1} * \cdots * \nu_{h_p}) \times \delta_{(h_1 + \cdots + h_p)}(g)$ is a Borel measurable function on G_2^p for each bounded Borel function g on $G_1 \oplus G_2$.

Let $f_j \in C_c(G_1 \oplus G_2)$ and put $f(x_1, \ldots, x_p) = f_1(x_1) \ldots f_p(x_p)$ for $(x_1, \ldots, x_p) \in (G_1 \oplus G_2)^p$. Then $(\lambda_{h_1} \times \cdots \times \lambda_{h_p})(f) = \lambda_{h_1}(f_1) \ldots \lambda_{h_p}(f_p)$ so that by (1)

(5) $(h_1, \ldots, h_p) \mapsto (\lambda_{h_1} \times \cdots \times \lambda_{h_p})(f)$ is Borel measurable.

Since $\{\sum_1^n f_{1i}(x_1) \ldots f_{pi}(x_p);\ f_{ji} \in C_c(G_1 \oplus G_2)\}$ is dense in $C_0((G_1 \oplus G_2)^p)$, (5) is Borel measurable for $f \in C_0((G_1 \oplus G_2)^p)$, hence, also for bounded Borel functions f on $(G_1 \oplus G_2)^p$. Let $\pi_p(x_1, \ldots, x_p) = x_1 + \cdots + x_p$ for $(x_1, \ldots, x_p) \in (G_1 \oplus G_2)^p$. Then for bounded Borel g on $G_1 \oplus G_2$ we have $\lambda_{h_1} * \cdots * \lambda_{h_p}(g) = (\lambda_{h_1} \times \cdots \times \lambda_{h_p})(g \circ \pi_p)$ and the claim follows.

Hence, by Claim 2, we can define a measure ξ in $M(G_1 \oplus G_2)$ as follows

$$\xi(f) = \overbrace{\int_{G_2}\int_{G_2} \cdots \int_{G_2}} \left\{ (\nu_{h_1} * \nu_{h_2} * \cdots * \nu_{h_p}) \times \delta_{(h_1 + h_2 + \cdots + h_p)} \right\} \times (f)\, d\eta(h_1)\, d\eta(h_2) \ldots d\eta(h_p)$$

for $f \in C_0(G_1 \oplus G_2)$.

Claim 3. $\xi = \mu^p$.

Let (γ_1, γ_2) be in $\widehat{G_1 \oplus G_2}$. Then we have

$$\begin{aligned} \hat{\xi}(\gamma_1, \gamma_2) &= \prod_{i=1}^{p} \int_{G_2} \hat{\nu}_{h_i}(\gamma_1)(-h_i, \gamma_2)\, d\eta(h_i) \\ &= \{\hat{\mu}(\gamma_1, \gamma_2)\}^p = (\mu^p)\hat{}(\gamma_1, \gamma_2). \end{aligned}$$

Thus we have $\xi = \mu^p$.

Let E be a Borel measurable set in $G_1 \oplus G_2$ with $m_{G_1 \oplus G_2}(E) = 0$, where $m_{G_1 \oplus G_2}$

denotes the Haar measure on $G_1 \oplus G_2$. Then there exists a Borel measurable set F_2 in G_2 such that (i) $m_{G_2}(F_2) = 0$ and (ii) $m_{G_1}(E_y) = 0$ if $y \notin F_2$, where $E_y = \{x \in G_1;\ (x, y) \in E\}$. Let α_p be the homomorphism from G_2^p onto G_2 such that $\alpha_p(h_1, h_2, \ldots, h_p) = h_1 + h_2 + \cdots + h_p$. We note that $\nu_{h_1} * \nu_{h_2} * \cdots * \nu_{h_p}$ belongs to $L^1(G_1)$ for $(h_1, h_2, \ldots, h_p) \in K^p$ by Lemma 1 and

$$\eta^p(F_2) = (\overbrace{\eta \times \eta \times \cdots \times \eta})(\alpha_p^{-1}(F_2)).$$

Hence, by Claim 1 and Claim 3, we have

$$\begin{aligned}\mu^p(E) &= \int_{G_2}\int_{G_2} \cdots \int_{G_2} \{(\nu_{h_1} * \nu_{h_2} * \cdots * \nu_{h_p}) \times \delta_{(h_1+h_2+\cdots+h_p)}\} \\ &\qquad\qquad \times (\chi_E)\, d\eta(h_1)\, d\eta(h_2) \ldots d\eta(h_p) \\ &= \int_{K^p} \{\ ''\ \}\, d(\eta \times \eta \times \cdots \times \eta)(h_1, h_2, \ldots, h_p) \\ &= \int_{K^p \cap \alpha_p^{-1}(F_2)} \{\ ''\ \}\, d(\eta \times \eta \times \cdots \times \eta)(h_1, h_2, \ldots, h_p) \\ &\quad + \int_{K^p \setminus \alpha_p^{-1}(F_2)} \{\ ''\ \}\, d(\eta \times \eta \times \cdots \times \eta)(h_1, h_2, \ldots, h_p) = 0.\end{aligned}$$

Hence we have $\mu^p \in L^1(G_1 \oplus G_2)$. Thus $E_1 \times E_2$ is a small p set in $\widehat{G_1 \oplus G_2}$. Q.E.D.

Dr. S. Saeki kindly pointed out the following Lemma 2 to the author.

LEMMA 2. *Suppose the product set $E_1 \times E_2$ of a small p set E_1 in $\hat{G}_1$ and a Riesz set E_2 in $\hat{G}_2$ is a small p set in $\hat{G}_1 \oplus \hat{G}_2$ for all metrizable LCA groups G_1 and G_2. Then the product set $E_3 \times E_4$ of a small p set E_3 in $\hat{G}_3$ and a Riesz set E_4 in $\hat{G}_4$ is a small p set in $G_3 \oplus G_4$ for all LCA groups G_3 and G_4.*

PROOF. Suppose there exists a measure μ in $M_{E_3 \times E_4}(G_3 \oplus G_4)$ such that μ^p does not belong to $L^1(G_3 \oplus G_4)$. Then, by Corollary 3 of [2], there exists a measure σ in $M_o(G_3 \oplus G_4)$ such that

$$(\mu * \sigma)^p = \mu^p * \sigma^p \notin L^1(G_3 \oplus G_4). \tag{A}$$

Since $\sigma \in M_0(G_3 \oplus G_4)$, there exist open σ-compact subgroups Γ_i of $\hat{G}_i$ such that $\operatorname{supp}(\hat{\sigma}) \subset \Gamma_3 \times \Gamma_4$ $(i = 3, 4)$. On the other hand, $\hat{\Gamma}_3$ and $\hat{\Gamma}_4$ are metrizable LCA groups. Evidently $E_3 \cap \Gamma_3$ is a small p set in Γ_3 and $E_4 \cap \Gamma_4$ is a Riesz set in Γ_4. Hence, by the hypothesis of this Lemma, $(E_3 \cap \Gamma_3) \times (E_4 \cap \Gamma_4)$ is a small p set in $\Gamma_3 \oplus \Gamma_4$. Therefore, since $\sigma * \mu \in M_{(E_3 \cap \Gamma_3) \times (E_4 \cap \Gamma_4)}(G_3 \oplus G_4)$, we have $(\sigma * \mu)^p \in L^1(G_3 \oplus G_4)$.

This contradicts (A). Q.E.D.

LEMMA 3. *Let G_3 and G_4 be metrizable LCA groups. Let E_3 be a small p set in $\hat{G}_3$ and E_4 a Riesz set in $\hat{G}_4$. Then $E_3 \times E_4$ is a small p set in $\widehat{G_3 \oplus G_4}$.*

PROOF. Let μ be a measure in $M_{E_3 \times E_4}(G_3 \oplus G_4)$. Then there exist open metrizable σ-compact subgroups $G_1 \subset G_3$ and $G_2 \subset G_4$ such that $\operatorname{supp}(\mu)$ is contained in $G_1 \oplus G_2$. Let π_{G_2} be the projection from $G_1 \oplus G_2$ onto G_2. Put $\eta' = \pi_{G_2}(|\mu|)$. Then,

by disintegration theory, there exists a family $\{\lambda'_h\}_{h\in G_2}$ in $M(G_1 \oplus G_2)$ such that

(1) $h \mapsto \lambda'_h(f)$ is a Borel measurable function of h for each bounded Borel measurable function f on $G_1 \oplus G_2$,

(2) $\operatorname{supp}(\lambda'_h) \subset G_1 \times \{h\}$,

(3) $\|\lambda'_h\| \leq 1$ and

(4) $\mu(g) = \int_{G_2} \lambda'_h(g)\, d\eta'(h)$ for each bounded Borel measurable function g on $G_1 \oplus G_2$.

Since $G_1 \oplus G_2$ is σ-compact and metrizable, there exists a countable dense set $\mathcal{Q} = \{f_m\}$ in $C_0(G_1 \oplus G_2)$. For each $g \in C_0(G_1 \oplus G_2)$, we define a function $\tilde{g}$ in $C_0(G_3 \oplus G_4)$ by $\tilde{g}(x) = g(x)$ for $x \in G_1 \oplus G_2$ and $\tilde{g}(x) = 0$ for $x \notin G_1 \oplus G_2$. We define measures $\eta \in M(G_4)$ and $\lambda_h \in M(G_3 \oplus G_4)$ $(h \in G_4)$ as follows

$$\eta(F) = \eta'(G_2 \cap F) \text{ for a Borel measurable set } F \text{ in } G_4,$$

$$\lambda_h(F') = \begin{cases} \lambda'_h(F' \cap G_1 \oplus G_2) & \text{if } h \in G_2, \\ 0 & \text{if } h \in G_4 \setminus G_2, \end{cases} \tag{5}$$

for a Borel measurable set F' in $G_3 \oplus G_4$. Then we have the following

(6) $h \mapsto \lambda_h(f)$ is a Borel measurable function of h for each bounded Borel measurable function f on $G_3 \oplus G_4$,

(7) $\operatorname{supp}(\lambda_h) \subset G_1 \times \{h\} \subset G_3 \times \{h\}$,

(8) $\|\lambda_h\| \leq 1$,

(9) $\mu(g) = \int_{G_4} \lambda_h(g)\, d\eta(h)$ for each bounded Borel measurable function g on $G_3 \oplus G_4$.

From (7), we have $d\lambda_h(x, y) = d\nu_h(x) \times d\delta_h(y)$, where ν_h is a measure in $M(G_3)$ with $\operatorname{supp}(\nu_h) \subset G_1$. Noting that $\operatorname{supp}(\eta) \subset G_2$, we may apply Lusin's theorem and regularity of η to obtain for each positive integer n a compact subset K_n of $\operatorname{supp}(\eta)$ such that

(i) $\eta(G_2 \setminus K_n) < 1/n$,

(ii) $h \mapsto \lambda_h(\tilde{f}_m)$ is a continuous function on K_n for each $\tilde{f}_m \in \tilde{\mathcal{Q}} = \{\tilde{f};\, f \in \mathcal{Q}\}$, and

(iii) for each $x \in K_n$ and neighborhood V of x, $\eta(V \cap K_n) > 0$.

Since $\tilde{\mathcal{Q}}$ is dense in $C_0(G_1 \oplus G_2)^{\sim} = \{\tilde{f};\, f \in C_0(G_1 \oplus G_2)\}$ we may replace (ii) by

(ii)′ $h \mapsto \lambda_h(\tilde{f})$ is continuous on K_n for each $\tilde{f} \in C_0(G_1 \oplus G_2)^{\sim}$.

Claim 1. $\nu_h \in M_{E_3}(G_3)$ for $h \in K_n$.

Let $f \in L^1(\hat{G}_3)$ with $\operatorname{supp}(f) \subset E_3^c$. Then since $\hat{\mu}(\gamma_1, \gamma_2) = 0$ for $\gamma_1 \notin E_3$ we have

$$\begin{aligned} 0 &= \int_{\hat{G}_3} \hat{\mu}(\gamma_1, \gamma_2) f(\gamma_1)\, d\gamma_1 \\ &= \int_{\hat{G}_3} \int_{G_4} \int_{G_3} (-x, \gamma_1) d\nu_h(x)(-h, \gamma_2)\, d\eta(h) f(\gamma_1)\, d\gamma_1 \\ &= \int_{G_4} \int_{G_3} \hat{f}(x)\, d\nu_h(x)(-h, \gamma_2)\, d\eta(h) \\ &= \int_{G_4} \nu_h(\hat{f})(-h, \gamma_2)\, d\eta(h). \end{aligned} \tag{10}$$

Hence, for each $F \in L^1(\hat{G}_4)$ and $f \in L^1(\hat{G}_3)$ with $\operatorname{supp}(f) \subset E_3^c$ we have

$$0 = \int_{\hat{G}_4}\int_{G_4} \nu_h(\hat{f})(-h, \gamma_2)\, d\eta(h) F(\gamma_2)\, d\gamma_2$$

$$= \int_{G_4} \nu_h(\hat{f})\hat{F}(h)\, d\eta(h). \tag{11}$$

Since $L^1(\hat{G}_4)\hat{}$ is dense in $C_0(G_4)$, hence also in $L^1(\eta)$, (11) holds for all $F \in L^1(\eta)$. It follows from (6) that $h \mapsto \nu_h(\hat{f})$ is bounded and Borel measurable, hence in $L^\infty(\eta)$, and so by (11)

$$\nu_h(\hat{f}) = 0, \qquad \eta\text{-a.e.} \tag{12}$$

Let $\beta \in C_c(G_4)$ satisfy $\beta = 1$ on K_n, $\beta = 0$ off G_2, and set $g(x, y) = (f * m_{G_1^\perp})\hat{}(x)\beta(y)$, where $G_1^\perp$ is the annihilator of G_1. Then $g \in C_0(G_1 \oplus G_2)\check{}$ and, since $\operatorname{supp}(\nu_h) \subset G_1$, we have $\lambda_h(g) = \nu_h((f * m_{G_1^\perp})\hat{})\delta_h(\beta) = \nu_h(\hat{f})$ for each $h \in K_n$. Hence, $h \mapsto \nu_h(\hat{f})$ is continuous on K_n by (ii)′, and this together with (12) and (iii) shows that $\nu_h(\hat{f}) = 0$ for each $h \in K_n$. Thus, for $h \in K_n$, $0 = \nu_h(\hat{f}) = \int_{\hat{G}_3} \hat{\nu}_h(\gamma_1) f(\gamma_1)\, d\gamma_1$.

Since f is any function in $L^1(\hat{G}_3)$ with $\operatorname{supp}(f) \subset E_3^c$ we have $\hat{\nu}_h(\gamma_1) = 0$ on E_3^c, and the claim is established. Moreover, since $\eta(G_4 \setminus \bigcup_1^\infty K_n) = 0$ we have proved that $\nu_h \in M_{E_3}(G_3)$ a.a. $h(\eta)$.

Since E is a Riesz set, we may prove that $\eta \in L^1(G_4)$ by arguing as in Theorem 1.

Claim 2. $(h_1, h_2, \ldots, h_p) \mapsto \{(\nu_{h_1} * \nu_{h_2} * \cdots * \nu_{h_p}) \times \delta_{(h_1+h_2+\cdots+h_p)}\}(g)$ is a Borel measurable function on $(G_4)^p$ for each $g \in C_0(G_3 \oplus G_4)$.

Indeed, since $\nu_h = 0$ if $h \notin G_2$, we have $(\nu_{h_1} * \nu_{h_2} * \cdots * \nu_{h_p}) \times \delta_{(h_1+h_2+\cdots+h_p)} = 0$ for $(h_1, h_2, \ldots, h_p) \notin (G_2)^p$.

On the other hand, $\lambda_n = \nu_h \times \delta_h$ may be regarded as a measure in $M(G_1 \oplus G_2)$.

Since G_1 and G_2 are σ-compact metrizable LCA groups, we may prove Claim 2 by arguing as in the proof of Claim 2, Theorem 1.

We now define a measure ξ in $M(G_3 \oplus G_4)$ as follows

$$\xi(f) = \int_{G_4} \cdots \int_{G_4} \{(\nu_{h_1} * \cdots * \nu_{h_p}) \times \delta_{(h_1+\cdots+h_p)}\}(f)\, d\eta(h_1) \ldots d\eta(h_p)$$

for $f \in C_0(G_3 \oplus G_4)$.

Then we have $\xi = \mu^p$. Let E_0 be a Borel measurable set in $G_3 \oplus G_4$ with $m_{G_3 \oplus G_4}(E_0) = 0$. Put $E = E_0 \cap G_1 \oplus G_2$. Since $\operatorname{supp}(\mu^p) \subset G_1 \oplus G_2$, we have $\mu^p(E_0) = \mu^p(E)$. Moreover, $m_{G_1 \oplus G_2}(E) = 0$. Thus we can prove that $\mu^p(E_0) = 0$ by using the techniques employed in Theorem 1. That is $\mu^p \in L^1(G_3 \oplus G_4)$. This completes the proof.

From Lemma 2 and Lemma 3, we obtain the following main theorem.

THEOREM 2. *Let G_1 and G_2 be LCA groups. Let E_1 be a small p set in $\hat{G}_1$ and E_2 a Riesz set in $\hat{G}_2$. Then $E_1 \times E_2$ is a small p set in $\widehat{G_1 \oplus G_2}$.*

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