THE FIRST EIGENVALUE OF THE LAPLACIAN FOR PLANE DOMAINS

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Abstract. We prove an improved lower bound for the first eigenvalue of the Laplacian of a connected plane domain in terms of its inradius and connectivity.

In this note we find a lower bound for the first eigenvalue, \( \lambda_1 \), of the Dirichlet problem for the Laplacian of a connected plane domain \( D \) in terms of its inradius \( \rho \) (the radius of the largest disk contained in \( D \)), and the connectivity \( k \) (i.e., the number of boundary components).

Hayman [3] was the first to prove an inequality of this type. He proved \( \lambda_1(D) > 1/900\rho^2 \) in the case \( k = 1 \) (i.e., \( D \) simply connected). Next, Osserman [4] showed that

\[
\lambda_1(D) > \frac{1}{4\rho^2}, \quad k = 1, 2,
\]

\[
\lambda_1(D) > \frac{1}{k^2\rho^2}, \quad k > 2.
\]

In that paper Osserman suggests that one might be able to get a bound of the form \( \lambda_1(D) > c/k\rho^2 \). Taylor [6] proved that such a bound exists, although no explicit constant \( c \) was given. Recently Cheng [2] has shown, using a completely different method, that \( c \) can be taken to be \( 1/(14000\pi)^2 \).

In this note we show

\[
\lambda_1(D) > \frac{1}{2k\rho^2} \quad \text{for } k > 2 \quad \text{(i.e., } c = \frac{1}{2}\text{)}.
\]

The method used here is similar to the one used in [4]. The method is useful only for plane domains, whereas Taylor’s, Hayman’s, and Cheng’s methods are useful in higher dimensions or for variable curvatures.

Theorem. Let \( D \) be a connected, \( k \)-connected, domain in the plane with inradius \( \rho \). Let \( A \) represent the area of \( D \) and \( L \) represent the total boundary length. Then

\[
\frac{L}{A} > \frac{1}{\rho}, \quad k = 1, 2. \quad (1)
\]

\[
\frac{L}{A} > \frac{2}{(1 + \sqrt{k - 1})\rho} > \frac{\sqrt{2}}{\sqrt{k} \rho}, \quad k > 2. \quad (2)
\]

\[
\lambda_1(D) > \frac{1}{4\rho^2}, \quad k = 1, 2. \quad (3)
\]

\[
\lambda_1(D) > \frac{1}{(1 + \sqrt{k - 1})^2\rho^2 > 1/2k\rho^2}, \quad k > 2. \quad (4)
\]

Received by the editors March 24, 1980. 
1980 Mathematics Subject Classification. Primary 52A40.

1Supported in part by NSF Grant MCS 76-01692.

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Proof. Inequalities (1) and (2) (applied to subdomains of $D$) imply inequalities (3) and (4) by Cheeger's result [1] as modified by Osserman [4]. Inequality (1) (and (3)) were proved in [4]. It is a simple computation to see $\sqrt{2} \sqrt{k} > (1 + \sqrt{k - 1} ) p$. Hence we need only show that for $k > 2$, $L/A > 2/(1 + \sqrt{k - 1} ) p$.

We consider two cases.

Case 1. $A < (1 + \sqrt{k - 1} )^2 \pi \rho^2$.

By the isoperimetric inequality for plane domains we have $L^2/A > 4 \pi$. Hence

$L/A > 2\sqrt{\pi}/\sqrt{A} > 2/(1 + \sqrt{k - 1} ) p$.

Case 2. $A > (1 + \sqrt{k - 1} )^2 \pi \rho^2$.

By a standard argument in the plane [4, p. 548] one has

$$\frac{\rho L}{A} > 1 - \frac{\pi (k - 2) \rho^2}{A} > 1 - \frac{(k - 2)}{(1 + \sqrt{k - 1} )^2} = \frac{2}{(1 + \sqrt{k - 1} )^2}.$$

This proves the theorem.

Remarks. Inequalities (1) and (2) are sharp for $k = 2$, with equality for a circular annulus. For $k = 1$ inequality (1) is strict but it is also the best possible, as was noted by Santaló [5, p. 155], as one sees by considering long thin rectangles. For $k > 3$ inequality (2) is strict and not the best possible. In this case one could ask for the best constants $C(k)$ such that $L/A > C(k)/\sqrt{k} \rho$. The theorem gives $C(k) > 2 \sqrt{k} / (1 + \sqrt{k - 1} )$, thus asymptotically $C(k) > 2$ (i.e. for every $\epsilon > 0$ and for sufficiently large $k$, $C(k) > 2 - \epsilon$). By considering large disks with trianularly packed points removed, one can see that asymptotically $C(k) < 2 \sqrt{2 \pi / 3 \sqrt{3}}$. Thus the theorem gives an estimate which is sharp for $k = 2$ and close to the best asymptotically.

As for inequality (4), Osserman [4, p. 552] gives examples where $\lambda_1(D) < \pi^2/\rho^2$; thus the inequality is not too far from the best possible.

References


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