A REMARK ON ANALYTIC SETS WITH 
σ-COMPACT SECTIONS

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Abstract. We show that there exists an analytic set \( A \subset \omega^n \times \omega^m \) and having \( \sigma \)-compact vertical sections such that \( A \) contains no analytic set \( B \) with compact vertical sections and having the same projection to the first coordinate as \( A \). This answers a question of J. R. Steel.

The problem considered in this note has its genesis in the following question posed by C. Dellacherie: Suppose \( A \subset \omega^n \times \omega^m \) is analytic (\( \Sigma^1_1 \)) and has \( \sigma \)-compact vertical sections. Can \( A \) be written as \( \bigcup_{n \geq 0} A_n \), where each \( A_n \) is analytic and has compact vertical sections? In [5], J. R. Steel showed that Dellacherie's question admits a negative answer. Steel then formulated the following question: Suppose \( A \subset \omega^n \times \omega^m \) is analytic and has \( \sigma \)-compact vertical sections. Does there exist an analytic set \( B \) such that \( B \subset A \), \( B \) has compact vertical sections and \( \pi(B) = \pi(A) \), where \( \pi \) denotes projection to the first coordinate? We prove in this note that the answer to Steel's question is also negative. Specifically, we prove

Theorem. There exists a \( \Sigma^1_1 \) set \( A \subset \omega^n \times \omega^m \) having countable (and hence \( \sigma \)-compact) vertical sections such that whenever \( B \) is \( \Sigma^1_1 \), \( B \subset A \) and \( B \) has compact vertical sections, then \( \pi(B) \neq \pi(A) \).

Proof. We follow the notation and terminology of [2]. Fix a \( \Sigma^1_1 \) set \( R \subset \omega^n \times \omega^m \times \omega^m \) which is universal for \( \Sigma^1_1 \) subsets of \( \omega^n \times \omega^m \). By [2, 4D.2], fix a \( \Pi^1_1 \)-recursive partial function \( d: \omega \times \omega^m \to \omega^m \) which parametrizes points in \( A_r(\alpha) \cap \omega^m \).

Define
\[
P(\alpha, s) \leftrightarrow \text{Seq}(s) \land (\forall \beta)(R(\alpha, \alpha, \beta) \to (\exists i < 1h(s))
(d((s)\downarrow, \alpha) \land d((s)\downarrow, \alpha) = \beta)).
\]

Clearly \( P \) is \( \Pi^1_1 \). By the Easy Uniformization Theorem [2, 4B.4] there is a \( \Pi^1_1 \) set \( G \) which uniformizes \( P \). Let \( f: \omega^m \to \omega \) be the (partial) function whose graph is \( G \). Then \( f \) is a \( \Pi^1_1 \)-recursive partial function. Next define
\[
Q(\alpha, \beta) \leftrightarrow f(\alpha) \downarrow \land (\exists i)(i < 1h(f(\alpha)) \land d((f(\alpha))\downarrow, \alpha) \downarrow \land d((f(\alpha))\downarrow, \alpha) = \beta).
\]

Now check that
(a) \( Q \) is \( \Pi^1_1 \),
(b) \( Q_\alpha = \{ \beta: Q(\alpha, \beta) \} \) is finite, and
(c) if \( R_{\alpha, \alpha} = \{ \beta: R(\alpha, \alpha, \beta) \} \) is finite, then \( R_{\alpha, \alpha} \subset Q_\alpha \).

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The last fact is an easy consequence of the Effective Perfect Set Theorem [2, 4F.1].

Finally, define

\[ A(\alpha, \beta) \leftrightarrow (\exists n)((\forall i)(\beta(i) = n) \& \neg Q(\alpha, \beta)). \]

Then \( A \) is \( \Sigma^1_1 \) and has countable vertical sections. Moreover, since each \( Q_\alpha \) is finite, it follows that \( \pi(A) = \omega^\omega \). Suppose now that \( B \subset A \), \( B \) is \( \Sigma^1_1 \) and \( B \) has compact vertical sections. Find a code \( \alpha_0 \) for \( B \); that is, find \( \alpha_0 \) such that \( B = R_{\alpha_0} \) (= \( \{ (\alpha, \beta): R(\alpha_0, \alpha, \beta) \} \)). Since \( B_\alpha \subset A_\alpha \) and \( A_\alpha \) is discrete, it follows that \( B_\alpha \) is finite. In particular, \( B_{\alpha_0} = R_{\alpha_0, \alpha_0} \) is finite. So, by (c), \( B_{\alpha_0} \subset Q_{\alpha_0} \). But also \( B_{\alpha_0} \subset A_{\alpha_0} \subset \omega^\omega \setminus Q_{\alpha_0} \). It follows that \( B_{\alpha_0} = \emptyset \) and hence \( \pi(B) \neq \omega^\omega = \pi(A) \). This completes the proof.

We conclude with some remarks.

A. The above shows that the following weak 'reduction' property fails for analytic sets:

If \( A_n, n > 0, \) are analytic subsets of \( \omega^\omega \) with \( \bigcup_{n>0} A_n = \omega^\omega \), then there exist analytic sets \( B_n, n > 0, \) such that \( (\forall n)(B_n \subset A_n), \lim \sup B_n = \emptyset \) and \( \bigcup_{n>0} B_n = \omega^\omega \).

B. With the same notation as in the proof of the Theorem, define \( A^* = (\omega^\omega \times \omega^\omega) \setminus Q \). Then \( A^* \) is \( \Sigma^1_1 \) and its vertical sections, being cofinite, are of measure one under any continuous probability measure on \( \omega^\omega \), are comeager and are dense \( G_\delta \)'s. But, as is easy to check, \( A^* \) does not admit a Borel \( (\Delta^1_1) \) uniformization. This shows that the uniformization results of Blackwell and Ryll-Nardzewski [1], Sarbadhikari [3] and Srivastava [4] do not extend to analytic sets.

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REFERENCES


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