SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

A COUNTABLE $k$-SPACE THAT IS NOT STRATIFIABLE

L. FOGED

Abstract. We add a point to a countable metric space to obtain a regular Fréchet space which is not stratifiable.

Below we show that a countable, regular nonstratifiable example of van Douwen and Pol [1] may be constructed so as to make it a Fréchet space. Thus a countable, regular $k$-space need not be stratifiable, answering Michael's question in [2].

Following [1], we view a function $f: A \rightarrow B$ as the set $\{ \langle x, f(x) \rangle : x \in A \}$ and the restriction of a function $f$ to a subset $S$ of its domain is denoted by $f|S$.

Let $\{ D_a : a \in c \}$ be an almost disjoint family of infinite subsets of $\omega$ and let $\{ f_a : a \in c \}$ be the set of all functions from $\omega$ to $\omega$. Let $q$ be any point not in $\omega \times (\omega + 1)$ and let $\Delta = \{ q \} \cup [\omega \times (\omega + 1)]$. Make $\omega \times (\omega + 1)$, with the usual product topology, an open subspace of $\Delta$. For finite $F \subset c$ and for $n \in \omega$ define

$$U(F, n) = \{ q \} \cup \left[ \left( \omega \setminus \bigcup_{a \in F} D_a \right) \times (\omega + 1) \cup \bigcup_{a \in F} f_a \right] \setminus [n \times (\omega + 1)].$$

The $U(F, n)$'s form a neighborhood base for $q$, each member of which omits but finitely many elements of $f_a|D_a$ for each $a \in c$. Then $\Delta$ is regular but not stratifiable [1].

The space $\Delta$ is Fréchet if the almost disjoint family $\{ D_a : a \in c \}$ satisfies the following property.

(*) If $A \subset \omega$ is not contained in a finite union of the $D_a$'s, then there is an infinite subset $A'$ of $A$ so that $A' \cap D_a$ is finite for every $a \in c$.

To see this first note that members of $\omega \times (\omega + 1)$ have countable neighborhood bases. Next, let $S$ be a subset of $\omega \times (\omega + 1)$ so that $q \in \text{cl} S$, with $A$ the projection of $S$ onto $\omega$. If there is a finite subset $F$ of $c$ so that $A \subset \bigcup_{a \in F} D_a$, then for some $a \in F$ the set $S \cap (f_a|D_a)$ is infinite and, as we saw, every neighborhood of $q$ contains all but finitely many of its members. On the other hand, if $A$ cannot be covered by finitely many of the $D_a$'s, then (*) gives an infinite subset $A'$ of $A$ such that $A' \cap D_a$ is finite for all $a \in c$. Picking a member of $\{(n) \times (\omega + 1) \} \cap S$ for each $n \in A'$, yields a sequence which converges to $q$.  

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An almost disjoint family satisfying (⋆) may be constructed as follows. Denote the rationals and irrationals in [0, 1] by \( Q \) and \( P \) respectively, identifying \( \omega \) with \( Q \) and \( c \) with \( P \). For each \( p \in P \), choose a sequence \( D_p \) of rationals in [0, 1] converging to \( p \), taking care that \( Q = \bigcup_{p \in P} D_p \). Then \( \{D_p : p \in P\} \) satisfies (⋆).

For suppose that \( A \subseteq Q \) so that every infinite subset of \( A \) meets a \( D_p \) in an infinite set. Let \( F = \{ p \in P : A \cap D_p \text{ is infinite}\} \). \( F \) is a finite set, for otherwise we could find a sequence \( \{p_j : j < \omega\} \) in \( F \) converging to a member \( r \) of \([0, 1] \setminus \{p_j : j < \omega\} \), such that for every \( j < \omega \) the set \( A \cap D_{p_j} \) is infinite. Then picking for each \( j < \omega \) a \( q_j \in A \cap D_{p_j} \setminus (D_r \cup \{q_i : i < j\}) \) so that \( |q_j - p_j| < 1/j \) (letting \( D_r = \emptyset \) if \( r \in Q \)) would give an infinite subset \( \{q_j : j < \omega\} \) of \( A \) which meets every \( D_p \) in a finite set.

Further, \( A \setminus \bigcup_{p \in F} D_p \) is finite; otherwise \( A \setminus \bigcup_{p \in F} D_p \) would contain a non-trivial convergent sequence \( A' \), necessarily meeting every \( D_p \) in a finite set. It follows that \( A \) can be covered by a finite collection of \( D_p \)'s.

REFERENCES


Department of Mathematics, University of Texas, El Paso, Texas 79968