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SHORTER NOTES

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A COUNTABLE k-SPACE THAT IS NOT STRATIFIABLE

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ABSTRACT. We add a point to a countable metric space to obtain a regular Fréchet space which is not stratifiable.

Below we show that a countable, regular nonstratifiable example of van Douwen and Pol [1] may be constructed so as to make it a Fréchet space. Thus a countable, regular k-space need not be stratifiable, answering Michael's question in [2].

Following [1], we view a function $f: A \to B$ as the set $\{\langle x, f(x)\rangle : x \in A\}$ and the restriction of a function f to a subset S of its domain is denoted by $f|S$.

Let $\{D_\alpha : \alpha \in c\}$ be an almost disjoint family of infinite subsets of ω and let $\{f_\alpha : \alpha \in c\}$ be the set of all functions from ω to ω. Let q be any point not in $\omega \times (\omega + 1)$ and let $\Delta = \{q\} \cup [\omega \times (\omega + 1)]$. Make $\omega \times (\omega + 1)$, with the usual product topology, an open subspace of Δ. For finite $F \subset c$ and for $n \in \omega$ define

$$U(F, n) = \{q\} \cup \left[\left[\left(\omega \setminus \bigcup_{\alpha \in F} D_\alpha\right) \times (\omega + 1) \cup \bigcup_{\alpha \in F} f_\alpha | D_\alpha\right] \setminus [n \times (\omega + 1)]\right].$$

The $U(F, n)$'s form a neighborhood base for q, each member of which omits but finitely many elements of $f_\alpha | D_\alpha$ for each $\alpha \in c$. Then Δ is regular but not stratifiable [1].

The space Δ is Fréchet if the almost disjoint family $\{D_\alpha : \alpha \in c\}$ satisfies the following property.

(*) If $A \subset \omega$ is not contained in a finite union of the D_α's, then there is an infinite subset A' of A so that $A' \cap D_\alpha$ is finite for every $\alpha \in c$.

To see this first note that members of $\omega \times (\omega + 1)$ have countable neighborhood bases. Next, let S be a subset of $\omega \times (\omega + 1)$ so that $q \in \operatorname{cl} S$, with A the projection of S onto ω. If there is a finite subset F of c so that $A \subset \bigcup_{\alpha \in F} D_\alpha$, then for some $\alpha \in F$ the set $S \cap (f_\alpha | D_\alpha)$ is infinite and, as we saw, every neighborhood of q contains all but finitely many of its members. On the other hand, if A cannot be covered by finitely many of the D_α's, then (*) gives an infinite subset A' of A such that $A' \cap D_\alpha$ is finite for all $a \in c$. Picking a member of $[\{n\} \times (\omega + 1)] \cap S$ for each $n \in A'$, yields a sequence which converges to q.

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An almost disjoint family satisfying (*) may be constructed as follows. Denote the rationals and irrationals in [0, 1] by $\mathbf{Q}$ and $\mathbf{P}$ respectively, identifying ω with $\mathbf{Q}$ and c with $\mathbf{P}$. For each $p \in \mathbf{P}$, choose a sequence D_p of rationals in [0, 1] converging to p, taking care that $\mathbf{Q} = \bigcup_{p \in \mathbf{P}} D_p$. Then $\{D_p : p \in \mathbf{P}\}$ satisfies (*).

For suppose that $A \subset \mathbf{Q}$ so that every infinite subset of A meets a D_p in an infinite set. Let $F = \{p \in \mathbf{P}: A \cap D_p \text{ is infinite}\}$. F is a finite set, for otherwise we could find a sequence $\{p_j : j < \omega\}$ in F converging to a member r of $[0, 1] \setminus \{p_j : j < \omega\}$, such that for every $j < \omega$ the set $A \cap D_{p_j}$ is infinite. Then picking for each $j < \omega$ a $q_j \in A \cap D_{p_j} \setminus (D_r \cup \{q_i : i < j\})$ so that $|q_j - p_j| < 1/j$ (letting $D_r = \varnothing$ if $r \in \mathbf{Q}$) would give an infinite subset $\{q_j : j < \omega\}$ of A which meets every D_p in a finite set.

Further, $A \setminus \bigcup_{p \in F} D_p$ is finite; otherwise $A \setminus \bigcup_{p \in F} D_p$ would contain a nontrivial convergent sequence A', necessarily meeting every D_p in a finite set. It follows that A can be covered by a finite collection of D_p's.

References

1. E. van Douwen and R. Pol, *Countable spaces without extension properties*, Bull. Acad. Polon. Sci. **25** (1977), 987–991.

2. E. van Douwen, *Nonstratifiable regular quotients of separable stratifiable spaces*, Proc. Amer. Math. Soc. **52** (1975), 457–460.

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