ON STANDARD SUBGROUPS OF TYPE $^2E_6(2)$

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Abstract. The purpose of this paper is to close one of the last gaps in the classification of finite simple groups containing a standard subgroup. We prove that a simple group containing a standard subgroup of type $^2E_6(2)$ has to be isomorphic to $F_2$, the baby monster.

One of the remaining standard form problems is the classification of finite groups containing a standard subgroup of type $^2E_6(2)$. A quasi-simple group $A$ is said to be a standard subgroup in a group $G$ provided

(i) $N_G(A) = N_G(C_G(A))$,
(ii) $|C_G(A)|$ is even,
(iii) $|C_G(A) \cap C_G(A)^g|$ is odd for all $g \in G - N_G(A)$,
(iv) $[A, A^g] \neq 1$ for all $g \in G$.

The purpose of this paper is to handle the case $A / Z(A) \cong ^2E_6(2)$ and $|Z(A)|$. Further we may assume $m_2(C_G(A)) = 1$. Otherwise a result due to M. Aschbacher and G. Seitz [2] yields $A \leq G$. Furthermore a Sylow 2-subgroup of $C_G(A)$ is cyclic. Otherwise the classical involution theorem due to M. Aschbacher [1] yields $A \leq G$. The case $A / Z(A) \cong ^2E_6(2)$ and $|Z(A)|$ odd has been treated by G. Seitz in [8]. In this paper we prove

Theorem. Let $G$ be a finite group, $O(G) = 1$, and $A$ a standard subgroup in $G$ such that $A / Z(A) \cong ^2E_6(2)$ and $|Z(A)|$ is even. Then $\langle A^G \rangle = A$ or $\langle A^G \rangle \cong F_2$, the baby monster.

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For the properties of $^2E_6(2)$ used in this paper see [11] and [10]. For the remainder suppose $A \leq G$.

1. Preliminary results.

(1.1) Lemma. The group $X = ^2E_6(2)$ possesses only one class of elements $\omega$ of order five, $C_X(\omega) \cong Z_5 \times A_5$.

Proof. [11, Lemmas (6.10) and (6.2)].

(1.2) Lemma. Let $\nu$ be an element of order 11 in $X = ^2E_6(2)$. Then $|N_X(\langle \nu \rangle)| = 110$ and $|C_X(\nu)| = 22$.

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Proof. [11, Lemmas (7.8) and (6.5)].

(1.3) Lemma. Let \( z \) be a 2-central involution in \( X = \text{^2E}_6(2) \). Then
(i) \( C_X(z) \) is an extension of an extraspecial 2-group \( Q \) of width 10 with \( \text{PSU}_6(2) \).
(ii) All involutions of \( X \) are conjugated in \( X \) to elements of \( Q \).
(iii) Let \( T \) be a Sylow 2-subgroup of \( C_X(z) \). Then \( J(T/\langle z \rangle) = Q/\langle z \rangle \).
(iv) \( C_T(z)/Q \) contains exactly one elementary abelian subgroup of order \( 2^9 \).

Proof. (i) follows from [10, Lemma 2]; (ii) follows from [10, Lemma 3]. An easy computation using the tables given in [11, pp. 502–505] yields (iii) and (iv).

(1.4) Lemma. Let \( X \) be a 2-fold covering group of \( ^2E_6(2) \). If \( x \in X \) and \( x^2 \in Z(X) \) then \( x^2 = 1 \).

Proof. [11, p. 503].

(1.5) Lemma. Let \( G \) be a finite group containing an involution \( d \) such that \( C_G(d)/O(C_G(d)) \) is a 2-fold covering of \( ^2E_6(2) \). Let \( S \in \text{Syl}_2(C_G(d)) \). Then \( Z(S) \) is elementary abelian of order 4. Let \( B \) be a Sylow 2-subgroup of \( O_{2^2}(C_G(Z(S))) \) and \( g \) a 2-element normalizing \( B \) and acting trivially on \( C_G(Z(S))/O_{2^2}(C_G(Z(S))) \). Then \( [g, d] = 1 \) or
(i) \( \langle B, g \rangle = C \) is extraspecial of width 11 and
(ii) \( |N_G(C)/C_G(C)| = 2^{17} \cdot 3^6 \cdot 53 \cdot 7 \cdot 11 \cdot 23 \).

Proof. \( |Z(S)| = 4 \) follows from Lemmas (1.3) and (1.4). As in the proof of [11, Lemma (4.4)] the existence of a field automorphism of \( ^2E_6(2) \) is not used. Application of [11, Lemma (4.4)] proves (i). The same argument is possible for [11, Lemmas (4.6) and (4.7)]. This yields (ii).

2. Proof of the theorem.

(2.1) Lemma. A Sylow 2-subgroup of \( C_G(A) \) is of order 2.

Proof. Let \( T_1 \in \text{Syl}_2(A) \). Then \( Z(T_1) = \langle z, d \rangle \) is of order 4, \( \langle d \rangle \in \text{Syl}_2(Z(A)) \). Let \( Y \) be the preimage of \( J(T_1/Z(T_1)) \). Then \( Y' = \langle z \rangle \), by Lemma (1.3). Let \( T \in \text{Syl}_2(N_G(A)) \). Then \( Z(T) = \langle z, S \rangle, S \leq C_G(A) \). Furthermore \( \langle z \rangle \) char \( T \). Let \( S_1 = T \cap C_G(A) \) and \( Y_1 \) the preimage of \( J(T/Z(T)) \). Then \( Z(Y_1) = \langle z, S_1 \rangle \).

Assume \( |S_1| > 4 \). Then \( T \in \text{Syl}_2(G) \). By Lemmas (1.3)(ii) and (1.4) each involution of \( AC_G(A) \) is conjugate in \( A \) to an involution in \( Y_1 \). Let \( y \in Y_1 \), \( d \neq y \sim d \) in \( G \). Then \( |Y_1 : C_{Y_1}(y)| = 2 \). Let \( T_2 = C_{T_1}(y) \in \text{Syl}_2(C_{N_G(A)}(y)) \) and \( T_3 \subseteq C_G(y) \). \( |T_3 : T_2| = 2 \). As \( \Omega_1(\phi(Z(C_{Y_1}(y)))) = \langle d \rangle \), \( C_{T_1}(y) \leq T_3 \). Choose \( t \in T_3 - T_2 \). As \( y \sim yz \) and \( d \sim dx \) in \( G \) we get \( z^t \neq z \), otherwise \( \langle z, y \rangle = \Omega_1(Z(T_3)) \sim \Omega_1(Z(T)) = \langle z, d \rangle \). Hence \( C_{T_1}(y) \cap C_{Y_1}(y) \) has to be elementary abelian. But this yields that \( C_{Y_1}(y)^{C_{Y_1}(y)} / C_{Y_1}(y) \) is abelian of rank at least 10. This contradicts Lemma (1.3)(iv) and the structure of \( \text{Out}(^2E_6(2)) \) [9]. Thus \( d^G \cap AC_G(A) = d \). Let \( y \) be an involution in \( N_G(A) - C_G(A) \). Then there is a fours group \( V \) contained in \( A \) such that \( V_y \leq y^G \). Let \( g \in G \) with \( y^g = d \). Then \( V^g \cap AC_G(A) \neq 1 \). But this contradicts \( d^G \cap AC_G(A) = d \). Thus \( d^G \cap N_G(A) = d \). Now application of [3] yields the contradiction \( A \lhd G \).
(2.2) Lemma. \(|G : N_G(A)|\) is even.

Proof. Suppose \(|G : N_G(A)|\) to be odd. Let \(T \in \text{Syl}_2(N_G(A))\). Then \(Z(T) = \langle d, z \rangle, \langle d \rangle \in \text{Syl}_2(Z(A))\). Let \(E\) be the preimage of \(J(T/Z(T))\). By Lemma (1.3) we have that \(E\) is the direct product of \(\langle d \rangle\) with an extraspecial group of width 10. Furthermore \(E' = \langle z \rangle\). By Lemma (1.3)(ii) every involution of \(A\) is conjugate in \(A\) to an involution of \(E\). Let \(y \neq d\) be an involution in \(E\) such that \(y \sim d\) in \(G\). As \(d \sim z \sim dz \sim y\) in \(G\). Further \(|E : C_E(y)| = 2\). Set \(T_1 = C_T(y)\). We may assume \(T_1 \in \text{Syl}_2(C_{N_G(A)}(y))\). Let \(T_2 < C_T(y)\), \(|T_2 : T_1| = 2\) and \(x \in T_2 - T_1\). Then \(z^x \neq z\), otherwise \(\langle z, y \rangle \sim \langle z, d \rangle\). Hence \(C_E(y)^x \neq C_E(y)\). Thus \(|C_E(y)^x C_E(y)/C_E(y)| > 2^9\). Now Lemma (1.3)(iv) yields that \(C_E(y)^x C_E(y)/C_E(y)\) is uniquely determined. But then (see [7, Lemma 1]) \(C_E(C_E(y)^x)\) contains only involutions conjugate to \(d, z\) or \(dz\) in \(A\). This is a contradiction. Hence \(d^G \cap C_G(A)A = d\).

Suppose \(y \in N_G(A) - A, y \sim d\) in \(G\). Then \(A\) contains a fours group \(V\) such that \(Vy < y^G\). But this contradicts \(d^G \cap C_G(A)A = d\). Thus \(d^G \cap N_G(A) = d\). Now the application of [3] yields the contradiction \(A \nsubseteq G\).

(2.3) Lemma. There are involutions in \(N_G(A)\) acting as field automorphisms on \(A/Z(A)\). In particular \(|N_G(A) : C_G(A)A| = 2\).

Proof. According to Lemma (2.2) \(d \sim dz\) in \(G\). Thus there is a 2-element \(g\) in \(N_G(\langle d, z \rangle)\) with \(d^g = dz\). Suppose \(N_G(A) = C_G(A)A\). By Lemma (1.3)(i) \(N_{N_G(A)}(\langle d, z \rangle)/O_{2^2}(N_{N_G(A)}(\langle d, z \rangle)) = PSU_2(2)\).

Assume

\[ [g, N_{N_G(A)}(\langle d, z \rangle)] \subseteq O_{2^2}(N_{N_G(A)}(\langle d, z \rangle)). \]

Set \(C = \langle O_2(C_G(\langle d, z \rangle)), g \rangle\). Then Lemma (1.5) yields that \(C\) is extraspecial of width 11 and \(|N_G(C)/C_G(C)C| = 2^{17} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23\). Let \(t\) be a 2-central involution in \(C_G(\langle d, z \rangle)/O_2(C_G(\langle d, z \rangle))\). Set \(N_G(C) = N_G(C)/C_G(C)C\) and \(\tilde{C} = C/C/\langle z \rangle\). Now \(\tilde{X} = O_2(C_{N_G(A)}(\langle d, z \rangle))/X = U_4(2);\) Lemma (1.3)(i). The group \(\tilde{X}\) centralizes in \(\tilde{C}\) a subgroup of order 27, see [11, Lemma (5.1)]. As \(C_{N_G(A)}(\langle d, z \rangle)/X = U_4(2)\) induces on \((\tilde{Y}/\langle \tilde{d} \rangle)^2\) two orbits of length 27 and 36, we get that \(\tilde{d}\) possesses exactly 28 conjugates under the action of \(N_{N_G(C)}(\tilde{X})\). Thus \(N_{N_G(C)}(\tilde{X})/\tilde{X}\) is of order \(2^8 \cdot 3^4 \cdot 5 \cdot 7\). As \(U_4(2)\) is involved we get that this group has to be simple. Checking the list of groups in [5] we get a contradiction. Thus we have that \(g\) induces an outer automorphism on \(N_{N_G(A)}(\langle d, z \rangle)/O_{2^2}(N_{N_G(A)}(\langle d, z \rangle))\). Then by Lemma (1.3)(i) \(O_2(N_{N_G(A)}(\langle d, z \rangle))/\langle z \rangle\) is the unique elementary abelian subgroup of order \(2^{21}\) in a Sylow 2-subgroup of \(N_G(\langle d, z \rangle)/\langle z \rangle\). Hence \(d^{C_G(z)} \cap O_2(N_{N_G(A)}(\langle d, z \rangle)) = \{d, dz\}\).

Assume now \(O(N_G(A)) \neq 1\). Let \(T\) be a Sylow 2-subgroup of \(A\) containing \(z\). Then \(C_G(T) = O(N_G(A))(z, d)\). Thus \(d \sim dz\) in \(N_G(O(N_G(A)))\). Now by induction we get that \(AO(N_G(O(N_G(A))))\) is normal in \(N_G(O(N_G(A)))\). But this is a contradiction. Thus \(O(N_G(A)) = 1\).

Let \(\omega\) be an element of order five in \(C_G(z) \cap N_G(A)\). By Lemma (1.1) we may
assume \( d \sim dz \) in \( C_G(\omega) \). By Lemmas (1.1) and (1.4) we have \( C_\omega(\omega)/\langle \omega \rangle \cong \langle d \rangle \times A_8 \). Let \( S \) be a Sylow 2-subgroup of \( C_G(\omega) \). Then \( S \) contains an elementary abelian subgroup \( B \) of order 32. Because of \( d^{C_\omega} \cap O_2(N_A(\langle d, z \rangle)) = \{d, dz\} \) we get that \( d \) has exactly 10 conjugates under \( N_{C_G(\omega)}(B) \). As \( A_8 \) contains no subgroup isomorphic to \( \Sigma_3 \times \Sigma_3 \) we get \( O_3(N_{C_G(\omega)}(B)/C_{C_G(\omega)}(B)) \) is nontrivial. But then we get the contradiction that \( d \) is weakly closed in \( B \) with respect to \( N_{C_G(\omega)}(B) \).

Hence we have shown that \( |N_G(A) : C_G(\omega)A| = 2 \). Then we may assume that \( [g, N_{C_G(\omega)}(\langle d, z \rangle)] \subseteq O_2(N_{C_G(\omega)}(\langle d, z \rangle)) \), as \( \text{Out}(PSU_6(2)) \cong \Sigma_3 \), see [9]. By Lemma (1.2) we may assume that \( g \) centralizes an element \( v \) in \( N_{C_G(\omega)}(\langle d, z \rangle) \) with \( v^{11} \in O(C_G(A)) \). By Lemma (1.2) we get that a Sylow 2-subgroup \( S_1 \) of \( N_{C_G(\langle v \rangle)}(\langle d, z \rangle) \) is of order eight. Clearly \( S_1 \) is abelian. We may assume that \( g \) normalizes \( S_1 \). Thus \( d \notin \phi(S_1) \). Then \( S_1 \) has to be elementary abelian. Thus there is an involution in \( N_{C_G(\omega)}(A) \) — \( C_G(A) \) normalizing a subgroup of order 11 in \( A \). Now the structure of \( \text{Aut}(2^2E_6(2)) \) [9] yields that this involution induces a field automorphism on \( A/Z(A) \).

**(2.4) Lemma.** We have \( \langle A G \rangle \cong F_2 \).

**Proof.** By Lemma (2.3) and [11] it is enough to show \( O(N_G(A)) = 1 \). Suppose \( K = O(N_G(A)) \neq 1 \). Let \( C = F^*(C_\omega(\langle d, z \rangle)) \). As \( d \sim dz \) in \( G \), \( d \sim dz \) in \( N_G(C) \). We have \( C_G(C) = K(d, z) \). Thus \( d \sim dz \) in \( N_G(K) \). As \( A < N_G(K) \) we get by induction \( N_G(K)/O(N_G(K)) = F_2 \) and \( [K, \langle A^{N_G(K)} \rangle] = 1 \). Let \( Y \) be a Sylow 2-subgroup of \( F^*(C_{N_G(K)}(z)) \). Then \( Y \) is extraspecial of width 11. The conjugacy classes of involutions in \( N_{N_G(K)}(Y) \) — \( Y \) are listed in [11, Table VI]. Let \( x \) be such an involution. Then it is easy to see that \( C_{N_{N_G(K)}(Y)}(T) = Z(T) \times K \), for \( T \in \text{Syl}_2(C_{N_{N_G(K)}(Y)}(x)) \). Thus \( d \sim x \) in \( C_G(z) \). Thus the weak closure of \( d \) in \( N_G(Y) \) with respect to \( C_G(z) \) is contained in \( Y \). Then \( Y \) is strongly closed in \( N_G(Y) \) with respect to \( C_G(z) \). Application of [4] yields now \( C_G(z) \subseteq N_G(K) \). Now the structure of centralizers of involutions in \( N_G(K) \) yields that \( N_G(K) \) controls \( G \)-fusion of 2-central involutions in \( N_G(K) \). But then \( G = N_G(K) \) by Holt’s theorem [6]. This contradicts \( O(G) = 1 \). The lemma is proved.

**References**

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