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A NOTE ON SINGULAR INTEGRALS WITH WEIGHTS

DOUGLAS S. KURTZ AND RICHARD L. WHEEDEN[1]

ABSTRACT. We prove two results concerning the behavior on weighted L^p spaces of Calderón-Zygmund singular integrals formed with kernels whose L^1 moduli of continuity satisfy the Dini condition. We also prove a result about the behavior of multiplier operators.

Let $K(x)$, $x \in \mathbf{R}^n$, be homogeneous of degree $-n$, have integral over $|x| = 1$ equal to zero,and satisfy the L^1-Dini condition

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < +\infty, \quad \text{where } \omega(\delta) = \sup \int_{|x'|=1} |K(\rho x') - K(x')|\, dx', \tag{1}$$

the sup being taken over all rotations ρ of the unit sphere with magnitude at most δ. Let $Tf(x) = \text{p.v. } (f * K)(x)$ denote the corresponding Calderón-Zygmund singular integral. Under the mild restriction (1), very little is known about the classes of nonnegative weight functions $w(x)$ for which either

$$\left(\int_{\mathbf{R}^n} |Tf(x)|^p w(x)\, dx\right)^{1/p} \leq c\left(\int_{\mathbf{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p}, \qquad 1 < p < \infty, \tag{2}$$

or

$$\int_{\{x:\, |Tf(x)| > \lambda\}} w(x)\, dx \leq \frac{c}{\lambda} \int_{\mathbf{R}^n} |f(x)| w(x)\, dx, \qquad \lambda > 0, p = 1, \tag{3}$$

where c is a constant independent of f and λ. Most such known results rely on stronger assumptions about K; see, e.g., [5]. If K only satisfies (1), both (2) and (3) are of course valid if $w(x) \equiv 1$ (see [1]); in this case the full strength of (1) is used only to obtain (3), (2) being true under the weaker assumption that K is of class $L \log^+ L(|x| = 1)$ ([1], [2]). If $w(x) = |x|^\alpha$, a result in [7] states that if K is just in $L \log^+ L(|x| = 1)$, then (2) holds if $-1 < \alpha < p - 1$, but does not hold if $\alpha > p - 1$ or $\alpha < -1$. This range of α is considerably more restricted than the range $-n < \alpha < n(p - 1)$ for which $w(x) = |x|^\alpha$ satisfies the A_p condition

$$\left(\frac{1}{|Q|}\int_Q w(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\, dx\right)^{p-1} \leq c,$$

where Q is any cube in $\mathbf{R}^n$.

The main purpose of the note is to prove two facts about T when K satisfies (1). These are related to the result in [7] quoted above. We will show that if $p = 1$, an

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analogous weak-type result (3) holds if K satisfies the stronger assumption (1), and that (2) (or (3) when $p = 1$) fails for $\alpha > p - 1$ or $\alpha < -1$ even if K satisfies (1). Specifically, we will prove

THEOREM 1. *Let $K(x)$ be homogeneous of degree $-n$, have mean-value zero on the unit sphere, and satisfy* (1). *If $-1 < \alpha < 0$, then* (3) *holds with $w(x) = |x|^\alpha$.*

THEOREM 2. *There is a kernel K which satisfies the assumptions listed in Theorem* 1 *and for which* (2) *and* (3) *fail when $w(x) = |x|^\alpha$ with $\alpha > p - 1$ or $\alpha < -1$.*

We also prove a result of a different nature which is related to a theorem in [5]. There we consider multiplier operators T defined by $\widehat{Tf}(x) = m(x)\hat{f}(x)$, where m is a bounded function which satisfies

$$\sup_{R>0}\left(R^{s|\alpha|-n}\int_{R<|x|<2R}|D^\alpha m(x)|^s\,dx\right)^{1/s} < +\infty \tag{4}$$

for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers with length $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant l$, where l is a preassigned positive integer and $s \geqslant 1$. The collection of such m is denoted $M(s, l)$, and the result in [5] in question states that if $m \in M(s, l)$, $1 < s \leqslant 2$, $[n/s] < l < n$ and $n/l \leqslant p \leqslant (n/l)'$, then (2) is valid for $w(x) = |x|^\alpha$ if $-n < \alpha < n(p - 1)$. Since this is precisely the range of powers of $|x|$ which satisfy the A_p condition, it seemed plausible that (2) might hold for any w in A_p. This, however, is not the case.

THEOREM 3. *Let $1 < s \leqslant 2$, $[n/s] < l < n$ and $n/l \leqslant p \leqslant (n/l)'$. Then there exist w in A_p and m in $M(s, l)$ such that the multiplier operator defined by $\widehat{Tf} = m\hat{f}$ does not satisfy* (2).

The example we will give for Theorem 3 arose in discussions with R. Hunt, B. Muckenhoupt and W.-S. Young.

In what follows, we write $f \in L^p_w$ if the right side of (2) is finite, and use c to denote constants which may be different in different occurrences.

PROOF OF THEOREM 1.

LEMMA 1. *Let $\Omega(x)$ be homogeneous of degree zero and integrable on $|x| = 1$. If $\alpha > -1$ and $|t| \leqslant 2|y|$, then*

$$\int_{|x|<10|y|}|\Omega(x)|\,|x + t + y|^\alpha\,dx \leqslant c\|\Omega\|_{L^1(|x|=1)}|y|^{n+\alpha}.$$

This is a special case of Lemma 1 in [7].

In the next lemma, we use the known fact that (1) implies (in fact, by [3], is equivalent to)

$$\int_{|x|>2|y|}|K(x - y) - K(x)|\,dx \leqslant c, \qquad y \in \mathbf{R}^n. \tag{5}$$

LEMMA 2. *Let K satisfy* (1) *and $-1 < \alpha < 0$. There is a constant c so that if Q is a cube with center y_Q, then for $y \in Q$*

$$\int_{|x-y_Q|>2\operatorname{diam} Q}|K(x - y) - K(x - y_Q)|\,|x|^\alpha\,dx \leqslant c\frac{1}{|Q|}\int_Q|x|^\alpha\,dx.$$

PROOF. Let $d = \operatorname{diam} Q$. If $d > 2|y_Q|$, then

$$\frac{1}{|Q|}\int_Q |x|^\alpha\,dx \geqslant \frac{c}{d^n}\int_{|x|<d/4} |x|^\alpha\,dx \geqslant cd^\alpha.$$

Since $|x - y_Q| > 2d$ implies $|x| \geqslant |x - y_Q| - |y_Q| > 2d - d/2 > d$, using the fact that $\alpha \leqslant 0$ together with (5), we get

$$\int_{|x-y_Q|>2d} |K(x-y) - K(x-y_Q)|\,|x|^\alpha\,dx$$

$$\leqslant d^\alpha \int_{|x-y_Q|>2|y-y_Q|} |K(x-y) - K(x-y_Q)|\,dx \leqslant cd^\alpha.$$

If $d \leqslant 2|y_Q|$, then

$$\frac{1}{|Q|}\int_Q |x|^\alpha\,dx \geqslant \frac{c}{d^n}\int_{|x-y_Q|<d/4} |x|^\alpha\,dx \geqslant c|y_Q|^\alpha.$$

Also, if $y \in Q$,

$$\int_{|x-y_Q|>4|y_Q|} |K(x-y) - K(x-y_Q)|\,|x|^\alpha\,dx \leqslant |y_Q|^\alpha$$

$$\cdot \int_{|x-y_Q|>4|y_Q|} |K(x-y) - K(x-y_Q)|\,dx \leqslant c|y_Q|^\alpha,$$

by the reasoning above, since $4|y_Q| \geqslant 2d \geqslant 2|y - y_Q|, y \in Q$. Thus, we have only to show that

$$\int_{2d<|x-y_Q|<4|y_Q|} |K(x-y) - K(x-y_Q)|\,|x|^\alpha\,dx \leqslant c|y_Q|^\alpha, \qquad y \in Q,$$

or, by a change of variables, that

$$\Phi = \int_{2d<|x|<4|y_Q|} |K(x-\delta) - K(x)|\,|x + y_Q|^\alpha\,dx \leqslant c|y_Q|^\alpha$$

where $|\delta| \leqslant d \leqslant 2|y_Q|$.

Consider first the case $d/2 < |y_Q| \leqslant 4d$. Note that $\{x\colon 2d < |x + \delta| < 4|y_Q|\} \subset \{x\colon |y_Q|/4 < |x| < 6|y_Q|\}$. Thus, by a change of variables, we have

$$\Phi \leqslant \int_{2d<|x|<4|y_Q|} |K(x-\delta)|\,|x + y_Q|^\alpha\,dx + \int_{|y_Q|/2<|x|<4|y_Q|} |K(x)|\,|x + y_Q|^\alpha\,dx$$

$$\leqslant \int_{|y_Q|/4<|x|<6|y_Q|} |K(x)|\,|x + \delta + y_Q|^\alpha\,dx + \int_{|y_Q|/2<|x|<4|y_Q|} |K(x)|\,|x + y_Q|^\alpha\,dx.$$

By the homogeneity of K, $K(x) = \Omega(x')/|x|^n$ where $x' = x/|x|$. Therefore,

$$\Phi \leqslant \int_{|y_Q|/4<|x|<6|y_Q|} \frac{|\Omega(x')|}{|x|^n}|x + \delta + y_Q|^\alpha\,dx$$

$$+ \int_{|y_Q|/2<|x|<4|y_Q|} \frac{|\Omega(x')|}{|x|^n}|x + y_Q|^\alpha\,dx$$

$$\leqslant c|y_Q|^\alpha,$$

by factoring out $|x|^{-n} \leqslant c|y_Q|^{-n}$ and applying Lemma 1.

If $4d < |y_Q|$, we write

$$\Phi = \int_{2d<|x|<|y_Q|/2} |K(x-\delta) - K(x)|\,|x+y_Q|^\alpha\,dx$$
$$+ \int_{|y_Q|/2<|x|<4|y_Q|} |K(x-\delta) - K(x)|\,|x+y_Q|^\alpha\,dx.$$

Using the fact that $|x + y_Q|$ is equivalent to $|y_Q|$ for $|x| \leqslant |y_Q|/2$ and applying (5), we see the first term on the right is bounded by $c|y_Q|^\alpha$. We get the same estimate for the second term by repeating the previous argument and using the fact that $|\delta| < |y_Q|/4$.

To prove Theorem 1, let f^* denote the Hardy-Littlewood maximal function of f, $f \geqslant 0$. The procedure in [8, p. 19], allows us to decompose $\{x: f^*(x) > \lambda\}$ into a sequence of nonoverlapping cubes $\{Q_k\}$ such that $f(x) \leqslant \lambda$ for a.e. x outside $\bigcup Q_k$ and $\int_{Q_k} f(x)\,dx/|Q_k| \leqslant c\lambda$. The proof of Theorem 1 is now fairly standard and we shall be brief. Using the Q_k above, write $f = g + b$ where $g = f$ outside $\bigcup Q_k$ and $g = \int_{Q_k} f(x)\,dx/|Q_k|$ on Q_k. Applying Theorem 1 of [7] and the fact that $0 \leqslant g \leqslant \lambda$, we have

$$\int_{\{x:\,|Tg(x)|>\lambda\}} |x|^\alpha\,dx \leqslant \frac{c}{\lambda^2}\int_{\mathbf{R}^n} g(x)^2|x|^\alpha\,dx \leqslant \frac{c}{\lambda}\int_{\mathbf{R}^n} g(x)|x|^\alpha\,dx.$$

Since $-1 < \alpha < 0$, $|x|^\alpha$ satisfies the A_1 condition $\int_Q |x|^\alpha\,dx/|Q| \leqslant c\,\mathrm{ess}_Q\inf|x|^\alpha$. Hence, by the definition of g,

$$\int_{\mathbf{R}^n} g(x)|x|^\alpha\,dx = \int_{(\cup Q_k)'} f(x)|x|^\alpha\,dx + \sum_k \frac{1}{|Q_k|}\int_{Q_k} f(t)\,dt \int_{Q_k} |x|^\alpha\,dx$$
$$\leqslant c\int_{\mathbf{R}^n} f(x)|x|^\alpha\,dx.$$

Let Q_k^* be Q_k expanded concentrically twice. Then

$$\int_{\cup Q_k^*} |x|^\alpha\,dx \leqslant c\int_{\cup Q_k} |x|^\alpha\,dx \leqslant \frac{c}{\lambda}\int_{\mathbf{R}^n} f(x)|x|^\alpha\,dx,$$

the last inequality being a corollary of the weak-type result in [6] for the Hardy-Littlewood maximal function.

Finally, let y_k be the center of Q_k. Since $\int_{Q_k} b(y)\,dy = 0$,

$$\int_{\mathbf{R}^n - \cup Q_k^*} |(K * b)(x)|\,|x|^\alpha\,dx = \int_{\mathbf{R}^n - \cup Q_k^*} \left|\sum \int_{Q_k} K(x-y)b(y)\,dy\right| |x|^\alpha\,dx$$
$$= \int_{\mathbf{R}^n - \cup Q_k^*} \left|\sum \int_{Q_k} \{K(x-y) - K(x-y_k)\}b(y)\,dy\right| |x|^\alpha\,dx.$$

Changing the order of integration, applying Lemma 2 and the definition of A_1, we have

$$\begin{aligned}\int_{\mathbf{R}^n - \cup Q_k^*} &|(K * b)(x)|\,|x|^\alpha\,dx \\ &\leqslant \sum \int_{Q_k} |b(y)| \left(\int_{\mathbf{R}^n - Q_k^*} |K(x-y) - K(x-y_k)|\,|x|^\alpha\,dx \right) dy \\ &\leqslant \sum \int_{Q_k} |b(y)| \left(\frac{c}{|Q_k|} \int_{Q_k} |x|^\alpha\,dx \right) dy \\ &\leqslant c \sum \int_{Q_k} |b(y)|\,|y|^\alpha\,dy \leqslant c \int_{\mathbf{R}^n} f(y)|y|^\alpha\,dy.\end{aligned}$$

Theorem 1 now follows as usual.

PROOF OF THEOREM 2. Consider the case $n = 2$. Define $\Omega(\theta)$ of period 2π by

$$\Omega(\theta) = \begin{cases} 1/|\theta|\,|\log^3|\theta||, & |\theta| < 1/2, \\ -1/|\theta - \pi|\,|\log^3|\theta - \pi||, & |\theta - \pi| < 1/2, \\ 0, & \text{elsewhere in } |\theta| < \pi. \end{cases} \tag{6}$$

Clearly, Ω has mean-value zero. To see it satisfies (1), let $|\rho| \leqslant \delta$, $\delta < \frac{1}{8}$. By symmetry and the fact that $\Omega(\theta) = \Omega(\theta + \rho) = 0$ if $\frac{1}{2} + \delta < |\theta| < \pi - \frac{1}{2} - \delta$,

$$\begin{aligned}\int_{-\pi}^{\pi} |\Omega(\theta + \rho) - \Omega(\theta)|\,d\theta &= 2\int_{|\theta| < 1/2 + \delta} |\Omega(\theta + \rho) - \Omega(\theta)|\,d\theta \\ &\leqslant 4\int_{|\theta| < 3\delta} |\Omega(\theta)|\,d\theta + 2\int_{2\delta < |\theta| < 1/2 + \delta} |\Omega(\theta + \rho) - \Omega(\theta)|\,d\theta \\ &= D_1 + D_2.\end{aligned}$$

We have

$$D_1 = 8\int_0^{3\delta} \frac{d\theta}{\theta|\log^3\theta|} = \frac{c}{\log^2 3\delta},$$

$$D_2 \leqslant 2\int_{2\delta < |\theta| < 1/2 - \delta} |\Omega(\theta + \rho) - \Omega(\theta)|\,d\theta + 4\int_{1/2 - 2\delta < |\theta| < 1/2 + 2\delta} |\Omega(\theta)|\,d\theta.$$

Since δ is small, $\Omega(\theta)$ is bounded for $\frac{1}{2} - 2\delta < |\theta| < \frac{1}{2} + 2\delta$. Thus the second term in the estimate for D_2 is bounded by $c\delta$. In the first term, $|\theta + \rho|$ is equivalent to $|\theta|$ since $|\theta| > 2\delta$ and $|\rho| < \delta$. Hence, by the mean-value theorem, the first term is at most $c\delta \int_{2\delta < |\theta| < 1/2 - \delta} d\theta/\theta^2|\log^3|\theta|| \leqslant c\delta \int_\delta^{1/2} d\theta/\theta^2|\log^3\theta|$. This, however, is bounded by $c(\log\delta)^{-3}$, as we now show. In fact, changing variables, we see the estimate in question reduces to $\int_1^t (e^x/x^3)\,dx \leqslant ce^t/t^3$, $t > 1$. However,

$$\int_1^t \frac{e^x}{x^3}\,dx \leqslant e^{t/2}\int_1^{t/2} \frac{dx}{x^3} + \frac{1}{(t/2)^3}\int_{t/2}^t e^x\,dx \leqslant c\left[e^{t/2} + \frac{e^t - e^{t/2}}{t^3}\right] \leqslant c\frac{e^t}{t^3}.$$

Collecting estimates, we see that the L^1-modulus of continuity $\omega(\delta)$ of Ω satisfies $\omega(\delta) \leqslant c[\delta + (\log 3\delta)^{-2} + |\log\delta|^{-3}]$ for small δ, so that (1) holds.

Now, let $K(x) = \Omega(x')/|x|^2$ be defined by (6) and let $f(x) = \chi\,(|x| < 1)$, so that $f \in L^p_{|x|^\alpha}(\mathbf{R}^2)$ if $\alpha > -2$ and $1 \leqslant p < \infty$. Let S be the strip $S = \{x = (x_1, x_2) \in \mathbf{R}^2$: $x_1 \geqslant 10, |x_2| < \frac{1}{2}\}$. If $x \in S$ and $|y| < 1$, then $\Omega(x - y) \geqslant 0$. Therefore, for $x \in S$,

$$Tf(x) = \int_{|y|<1} \frac{\Omega(x-y)}{|x-y|^2}\,dy \geqslant \frac{c}{|x|^2}\int_{|y|<1}\Omega(x-y)\,dy = \frac{c}{|x|^2}\int_{|x-y|<1}\Omega(y)\,dy.$$

Hence, there is a constant c_1 so that

$$Tf(x) \geqslant \frac{c}{|x|^2}\int_{|x|-c_1}^{|x|+c_1}\rho\,d\rho\int_{|\theta|<c_1/|x|}\frac{d\theta}{|\theta|\,|\log|\theta||^3}, \qquad x \in S.$$

A simple computation gives $Tf(x) \geqslant c/|x|\log^2|x|$, $x \in S$. This implies

$$\int_{\mathbf{R}^2}|Tf(x)|^p|x|^\alpha\,dx \geqslant c\int_S \frac{|x|^\alpha\,dx}{|x|^p\log^{2p}|x|}$$
$$\geqslant c\int_{10}^{\infty}\rho^{\alpha-p+1}(\log\rho)^{-2p}\left(\int_0^{c/\rho}d\theta\right)d\rho = c\int_{10}^{\infty}\rho^{\alpha-p}(\log\rho)^{-2p}\,d\rho.$$

The last integral is $+\infty$ if $\alpha > p - 1$. This proves the part of Theorem 2 for $\alpha > p - 1, p > 1$. The part for $\alpha < -1, p > 1$, follows by duality.

If $p = 1$, the estimates above show that $\{x\colon |Tf(x)| > \lambda\}$ contains the part of S with $|x| < c_\beta\lambda^{-1/\beta}$ for any $\beta > 1$. It follows easily that the integral with respect to $|x|^\alpha\,dx$ over this set cannot be $O(\lambda^{-1})$ for small λ unless $\alpha \leqslant \beta - 1$. This proves the part of Theorem 2 for $p = 1$, $\alpha > 0$. For $p = 1$ and $\alpha < -1$, let f_N be the characteristic function of the part of S with $|x| < N$, N large. Computations like those above show that $\int_{\mathbf{R}^2}|f_N(x)|\,|x|^\alpha\,dx$ is bounded in N if $\alpha < -1$, and that $Tf_N(x) \geqslant c\log\log N$ for x in a small neighborhood of 0 independent of N. If we choose $\lambda = c\log\log N$, it follows that (3) cannot hold for $w(x) = |x|^\alpha$, $\alpha < -1$.

The same technique can be used in $\mathbf{R}^n$, $n \geqslant 2$, even for other kinds of Dini conditions. We say Ω satisfies the L^r-Dini condition, $1 \leqslant r \leqslant \infty$, if (1) holds with ω defined in terms of the $L^r(|x| = 1)$ norm. Fix ξ in $\mathbf{R}^n$ with $|\xi| = 1$, and for $|x| = 1$, define $\Omega_r(x) = |x - \xi|^{-(n-1)/r}|\log|x - \xi||^{-3}$ if $|x - \xi| < \frac{1}{2}$, $\Omega_r(x) = -|x + \xi|^{-(n-1)/r}|\log|x + \xi||^{-3}$ if $|x + \xi| < \frac{1}{2}$, and $\Omega_r(x) = 0$ otherwise. As before, Ω_r satisfies the L^r-Dini condition, but (2) fails for $f(x) = \chi\,(|x| < 1)$ if $w(x) = |x|^\alpha$, $\alpha > p - 1 + (n-1)p/r'$. If $p > 1$, duality shows (2) also fails if $\alpha < -1 - (n-1)p/r'$. As in [7], if $\Omega(x')$ is $+1$ for $x_n \geqslant 0$ and -1 for $x_n < 0$, Ω satisfies the L^r-Dini condition, $1 \leqslant r < \infty$, but (2) fails when $\alpha \geqslant n(p-1)$ or, by duality, $\alpha \leqslant -n$. These results thus agree with the negative conclusions of Theorem 1 of [7]. If $p = 1$, analogous facts again hold for (3).

Proof of Theorem 3. Fix s, l and p with $1 < s \leqslant 2$, $[n/s] < l < n$, l an integer, and $n/l \leqslant p \leqslant (n/l)'$. Since $l < n$ and $p > 1$, there exists $\beta > 0$ with $lp - n < \beta < n(p-1)$. Choose $\gamma < 0$ so that $-n < \gamma < \beta - lp$. (If $p = 2$, we may take $l < \beta = -\gamma < n$.) Let $|\eta| = 1$ and define $m(x) = e^{ix\cdot\eta}(1 + |x|^2)^{-l/2}$, $w(x) = |x|^\beta|x - \eta|^\gamma$. Then $m \in M(s, l)$, as was shown in [5], and $w \in A_p$, which follows either by direct computation or by appealing to the result of [4]. As in [5], if $\delta p > 1$, the function

$$f(x) = |x|^{-(n+\beta)/p}|\log|x||^{-\delta}\chi(\{x\colon |x| < \mu\})$$

is in L_w^p, and if $|x - \eta| < \mu/2$, $Tf(x) \geqslant c|x - \eta|^{l-(n+\beta)/p}|\log|x - \eta||^{-\delta}$ ($\mu > 0$ depends on l). Thus, $Tf \notin L_w^p$ since $\{l - (n + \beta)/p\}p + \delta = lp - n - \beta + \gamma < -n$.

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Department of Mathematics, Purdue University, West Layfayette, Indiana 47907

Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903