THE STRUCTURE OF \( \omega \)-LIMIT SETS OF NONEXPANSIVE MAPS

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Abstract. It is shown that \( \omega \)-limit sets of nonexpansive maps carry the structure of monothetic topological groups. This fact is then used to answer a question of Edelstein.

Let \( T: D \to D \) be a map from a subset \( D \) of a metric space \( X \) into itself. We call \( T \) nonexpansive if \( d(T(x), T(y)) < d(x, y) \) for all \( x \) and \( y \) in \( D \). For \( x \) in \( X \), we call the set \( \gamma(x) = \bigcup_{n \in \mathbb{Z}^+} T^n(x) \) the orbit of \( x \), while the (possibly empty) set \( \omega(x) = \{ y \in X: y = \lim_{N} T^n(x), \, N \text{ a strictly increasing sequence in } \mathbb{Z}^+ \} \) is called the \( \omega \)-limit set of \( x \). A subset \( \Omega \) of \( X \) is called minimal under \( T \) if \( \Omega = \overline{\gamma(y)} \) for each \( y \) in \( \Omega \), and is called strongly invariant under \( T \) if for each \( n \) in \( \mathbb{Z}^+ \) the map \( T^n \) is a homeomorphism of \( \Omega \) onto itself.

If \( T \) is a nonexpansive selfmap of the closed subset \( C \) of \( X \), then it is known from the theory of dynamical systems [4] that a nonempty \( \omega \)-limit set is actually minimal, and that the semigroup \( \{ T^n: n \in \mathbb{Z}^+ \} \) may be extended to a group of homeomorphisms \( \{ T^n: n \in \mathbb{Z} \} \) on this set. Further algebraic structure is provided by the following.

Theorem 1. Let \( C \) be a closed set in the Banach space \( X \) and let \( T: C \to C \) be nonexpansive. If for some \( x \in C \) the \( \omega \)-limit set \( \Omega \) of \( x \) is nonempty, then there exists a binary operation in \( \Omega \) under which it is a monothetic topological group in the topology induced by the metric of \( X \). (Recall a topological group \( G \) is called monothetic if it contains an element \( x \) such that \( \{ x^n: n \in \mathbb{Z} \} \) is dense in \( G \).

Proof. If \( G \) is nonempty, then as before it is minimal and strongly invariant under \( T \), and \( T^n \) is an isometric homeomorphism of \( G \) for all \( n \) (see [4, Theorem 1]). Choose \( e \in G \) arbitrarily and define a binary operation on \( G \) (denoted by juxtaposition) as follows: for any \( b \in G \), find a subsequence \( N_b \) of \( \mathbb{Z}^+ \) such that \( \lim_{N_b} T^n(e) = b \); then for \( a \in G \) define \( ab = \lim_{N_a} T^n(a) \). Using the fact that \( T \) is isometric on \( G \), it is straightforward to show that this limit exists and is unique. In fact, by applying a theorem of Moore [7, p. 100] on double sequences, it can be shown that this operation is abelian. Now the element \( e \) is clearly the identity, and since \( T \) is a homeomorphism on \( G \), inverse are given by the formula \( b^{-1} = \lim_{N_b} T^{-n}(e) \). Associativity follows directly from the definition. Thus \( G \) is a group.

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Suppose \( a_n \to a \), \( b_n \to b \) are sequences with all elements belonging to \( G \). Then
\[
\| a_n b_n^{-1} - ab^{-1} \| \leq \| b_n^{-1} a_n - b^{-1} a_n \| + \| a_n b_n^{-1} - ab^{-1} \| \\
= \lim_{N_n} T^n a_n - \lim_{N_n} T^n b_n^{-1} + \lim_{N_n} T^{-n} a_n - \lim_{N_n} T^{-n} b^{-1} \\
= \| b_n^{-1} - b^{-1} \| + \| a_n - a \|
\]
which is small for large enough \( n \). The operation is therefore continuous in both variables and \( G \) is a topological group.

Remarks. Della Riccia [5] has shown that if we assume only an equicontinuous semiflow acting on either a locally compact or a complete metric space, then again a nonempty \( \omega \)-limit set \( \omega(x) \) is minimal and the semiflow may be extended to an equicontinuous flow of homeomorphisms of \( \omega(x) \). Our result holds in this setting as well, since in the proof of Theorem 1, equicontinuity of \( T^n \) suffices to show existence and uniqueness of the limits. The remainder of the proof carries through directly.

It is known [8, p. 84] that if \( G \) is a locally compact monothetic group, then \( G \) is either compact or topologically equivalent to \( \mathbb{Z} \). Unfortunately, this result can fail outside the locally compact setting. Rolewicz [9] has described a monothetic complete metric abelian group which is neither discrete nor compact. One may also construct an example in \( L^2(\mu) \) where \( \mu \) is a nonatomic probability measure on the circle. Choose \( \mu \) so that \( \lim \sup |\hat{\mu}| = 1 \) and set \( Uf = e^{i\theta}f \) in \( L^2(\mu) \). Then \( \omega \)-limit sets of \( U \) have the desired properties.

Examples are also available from constructions of Edelstein [6]. He has given an example of a fixed point free isometry of Hilbert space which has a point whose \( \omega \)-limit set is nonempty and unbounded. The argument of the present paper shows this \( \omega \)-limit set must have been non-locally-compact.

However, we may use Theorem 1 to give an affirmative answer to a question posed by Edelstein [10, p. 66].

**Theorem 2.** Let \( X \) be a finite-dimensional, not necessarily strictly convex, Banach space, and let \( D \) be an arbitrary closed subset of \( X \). If \( \omega(x) \) is nonempty for some \( x \) in \( X \), then every orbit is bounded.

**Proof.** \( X \) is now locally compact, so \( G = \omega(x) \) is either compact or isomorphic to \( \mathbb{Z} \). But \( G \) is recurrent, so it must be compact, hence bounded. By nonexpansivity, each orbit is also bounded.

Remarks. We note that different arguments obtain this result in the strictly convex case [6]. If \( D \) is closed in \( \mathbb{R}^n \), and is in addition convex, then by standard arguments we easily obtain the existence of fixed points. Thus a fixed point free nonexpansive map on all of \( \mathbb{R}^n \) has the property that \( T^n(x) \to \infty \) for all \( x \). This formulation of the result is reminiscent of the Brouwer Translation Theorem: if \( T \) is a homeomorphism of \( E^2 \) onto itself which preserves orientation and has no fixed points, then \( T^n(x) \to \infty \) for all \( x \) as \( n \to \pm \infty \) (see [1] or [3]).

For \( D \) closed but not convex, the obvious example of a rotation of the circle shows that we cannot expect fixed points. If \( D \) is assumed topologically trivial,
however, one may still hope for a fixed point. We do not know if such a result holds, but can assert that it cannot be valid for topological reasons alone. For Conner and Floyd (see [2, p. 58]) have constructed a periodic transformation of \( \mathbb{R}^n \) which is fixed point free. For such a map an equivalent metric can be defined so that the map is a periodic isometry. Clearly this new metric cannot be a Banach space norm on \( \mathbb{R}^n \) since a nonempty convex invariant set could easily be produced.

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