ON NEAR-DERIVATIONS

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Abstract. In this note we show how near-derivations can be expressed by biadditive and additive functions satisfying further conditions.

1. The concept of near-derivation has been introduced and discussed by Lawrence, Mess and Zorzitto [3] in connection with nonnegative information functions. A real-valued function $\gamma$ defined on the reals $\mathbb{R}$ is called a near-derivation if

$$\gamma(xy) = xy\gamma(y) + y\gamma(x) \quad \text{for all } x, y \in \mathbb{R},$$

$$\gamma(x + y) > \gamma(x) + \gamma(y) \quad \text{for all } x > 0, y > 0,$$

$$\gamma(r) = 0 \quad \text{for all rational } r.$$  

Daróczy and Maksa showed in [1] that there exists a near-derivation which is not a derivation. Namely, if $d: \mathbb{R} \to \mathbb{R}$ is a nonidentically zero derivation then the function $\gamma$ defined by

$$\gamma(x) = \begin{cases} d(d(x)) - d(x)^2/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is such a near-derivation. In this note we present a method for recovering near-derivations in terms of biadditive functions on $\mathbb{R}^2$ and additive functions on $\mathbb{R}$ satisfying further conditions.

2. It has been proved in [3] that for any near-derivation $\gamma$ the finite limit

$$\alpha(x) = \lim_{n \to \infty} \gamma(x + n)$$

exists for all $x \in \mathbb{R}$. Furthermore, the function $\alpha: \mathbb{R} \to \mathbb{R}$ defined by (5) has the following properties:

(a) $\gamma(x) < \alpha(x)$ for all $x > 0$,
(b) $\alpha(x + y) = \alpha(x) + \alpha(y)$ for all $x, y \in \mathbb{R}$,
(c) $2x\alpha(x) < \alpha(x^2)$ for all $x \in \mathbb{R}$,
(d) $\alpha(r) = 0$ for all rational $r$.

Using (a) and (1)–(3) we have

$$\alpha(\frac{1}{t}) + \frac{1}{t} \alpha(t) = |t| \alpha\left(\frac{1}{|t|}\right) + \frac{1}{|t|} \alpha(|t|)$$

$$> |t| \gamma\left(\frac{1}{|t|}\right) + \frac{1}{|t|} \gamma(|t|) = \gamma(1) = 0$$

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that is,
\[(e) \ t\alpha(1/t) + (1/t)\alpha(t) > 0 \text{ for all } t \in \mathbb{R} \setminus \{0\}.
\]

Suppose that the function \( \alpha: \mathbb{R} \to \mathbb{R} \) satisfies (b)–(e) and define \( A: \mathbb{R}^2 \to \mathbb{R} \) by
\[
A(x,y) = \alpha(xy) - x\alpha(y) - y\alpha(x). \quad (6)
\]

It is easy to see that \( A \) has the properties:
\[
A(x,y) = A(y,x), \quad (7)
\]
\[
A(x + y, z) = A(x, z) + A(y, z), \quad (8)
\]
\[
A(x, x) > 0, \quad (9)
\]
\[
A(xy, z) + zA(x, y) = A(x, yz) + xA(y, z), \quad (10)
\]
\[
A(t, 1/t) < 0, \quad (11)
\]

for all \( x, y, z \in \mathbb{R} \) and \( t \in \mathbb{R} \setminus \{0\} \). It follows from [2] that a function \( A: \mathbb{R}^2 \to \mathbb{R} \) satisfying (7)–(11) is always of the form (6) where \( \alpha: \mathbb{R} \to \mathbb{R} \) has the properties (b)–(e).

**Theorem 1.** Suppose that the function \( \alpha: \mathbb{R} \to \mathbb{R} \) satisfies (b)–(e) and define \( A: \mathbb{R}^2 \to \mathbb{R} \) by (6). Then the function \( \gamma \) given by

\[
\gamma(x) = \begin{cases} 
\alpha(x) - \sum_{n=1}^{\infty} 2^{n-1}x^{1/2^n-1}A(x^{1/2^n}, x^{1/2^n}) & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-\gamma(-x) & \text{if } x < 0
\end{cases} \quad (12)
\]

is a near-derivation.

**Proof.** Let \( \gamma_n(x) = 2^{n-1}x^{1/2^n-1/2^n}\alpha(x^{1/2^n}) \) for \( x > 0 \) and \( n = 0, 1, \ldots \). Then
\[
\gamma_n(x) - \gamma_{n-1}(x) = -2^{n-1}x^{1/2^n-1}A(x^{1/2^n}, x^{1/2^n}). \quad (13)
\]

Hence, by (9), we get that the sequence \((\gamma_n(x))\) is decreasing for all fixed \( x > 0 \). According to (c) and (11)
\[
\gamma_n(x) + x^2\gamma_{n-1}(1/x) > -2^nxA(x^{1/2^n}, x^{-1/2^n}) > 0
\]

and therefore
\[
\gamma_n(x) > -x^2\gamma_{n-1}(1/x) > -x^2\gamma_0(1/x),
\]

which means that the sequence \((\gamma_n(x))\) is bounded below for all fixed \( x > 0 \). Thus \((\gamma_n(x))\) is convergent and its limit is in \( \mathbb{R} \). On the other hand (13) and (12) imply that
\[
\gamma(x) = \lim_{n \to \infty} \gamma_n(x) \quad (14)
\]

holds for all \( x > 0 \). By the Cauchy-Schwarz inequality for bilinear forms on a rational vector space
\[
|A(u, v)| \leq \sqrt{A(u, u)} \sqrt{A(v, v)} \quad (15)
\]
for all \( u, v \in \mathbb{R} \). Let \( x > 0, y > 0 \) and \( u = x^{1/2^n}, v = y^{1/2^n} \) \( (n = 0, 1, \ldots) \). Using (6), the definition of \((\gamma_n(x))\) and (13), (15) implies that
\[
|\gamma_n(xy) - x\gamma_n(y) - y\gamma_n(x)| \leq 2\sqrt{xy} \sqrt{\gamma_{n-1}(x) - \gamma_n(x)} \sqrt{\gamma_{n-1}(y) - \gamma_n(y)}.
\]
Hence, by (14), we obtain (1) for all $x > 0, y > 0$. From (12) and (9) $\gamma(t) < \alpha(t)$ for $t > 0$. Thus
\[
\gamma\left(\frac{x}{x+y}\right) + \gamma\left(\frac{y}{x+y}\right) < \alpha\left(\frac{x}{x+y}\right) + \alpha\left(\frac{y}{x+y}\right) = \alpha(1) = 0
\]
for all $x > 0, y > 0$. By (1) this implies (2) for $x > 0, y > 0$. To prove (3) let $r$ be a positive rational number. Then $A(r, r) = 0$ therefore by (15) $A(r, u) = 0$ for all $u \in \mathbb{R}$. Substituting $x = y = \sqrt{r}, z = 1/\sqrt{r}$ in (10) we see that $A(\sqrt{r}, \sqrt{r}) = rA(\sqrt{r}, 1/\sqrt{r})$. Thus (9) and (11) give that $A(\sqrt{r}, \sqrt{r}) = 0$. By induction we have $A(r^{1/2^n}, r^{1/2^n}) = 0$; thus (12) and (d) imply (3) for all positive rational $r$. Since $\gamma$ is an odd function the proof is complete.

**Theorem 2.** Let $\gamma$ be a near-derivation. Then there exist functions $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ and $A: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfying (b) and (7)–(11), respectively such that (12) holds for all $x \in \mathbb{R}$.

**Proof.** We have known that there exists a function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ with the properties (a)–(e). Thus the function $A: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by (6) satisfies (7)–(11). Define the function $\delta$ on $\mathbb{R}$ by
\[
\delta(x) = \begin{cases} 
\alpha(x) - \sum_{n=1}^{\infty} 2^{n-1} x^{1-1/2^{n-1}} A(x^{1/2^n}, x^{1/2^n}) & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-\delta(-x) & \text{if } x < 0.
\end{cases}
\]
Applying Theorem 1 we get that $\delta$ is a near-derivation. Using (6), (a) and (1) we have for $x > 0$
\[
\delta(x) = \lim_{n \to \infty} 2^n x^{1-1/2^n} \alpha(x^{1/2^n}) > \lim_{n \to \infty} 2^n x^{1-1/2^n} \gamma(x^{1/2^n}) = \gamma(x).
\]
Since $\delta - \gamma$ satisfies (1) this implies that $\delta = \gamma$, thus the proof is complete.

We remark that if $d$ is a nonidentically zero derivation and $\alpha(x) = d(d(x)), A(x, y) = 2d(x)d(y)$ $(x, y \in \mathbb{R})$ then Theorem 1 gives the example (4).

**References**


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