THE SIMILARITY PROBLEM FOR REPRESENTATIONS OF C*-ALGEBRAS

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Abstract. Let \( \pi: A \rightarrow B(H) \) be a bounded homomorphism of a C*-algebra into the bounded operators on a Hilbert space. We prove that, if \( \pi \) is cyclic, there is a *-representation \( \theta: A \rightarrow B(H) \) and a bounded one-to-one positive operator \( P \) such that \( P\theta(a) = \pi(a)P \). We include applications to \( \theta \)-derivations and invariant operator ranges for operator algebras.

1. Introduction. Let \( \pi: A \rightarrow B(H) \) be a continuous representation of a C*-algebra \( A \) into the bounded operators on a Hilbert space \( H \). In this paper, we are concerned with whether or not there exist a *-representation \( \theta: A \rightarrow B(H_0) \) and a bounded invertible operator \( S: H_0 \rightarrow H \) such that \( S\theta(a) = \pi(a)S \) for all \( a \in A \). That is, must \( \pi \) be similar to a *-representation? In the context of C*-algebras, the question was first raised by Kadison in [11]. It was shown in [5] that \( \pi \) is similar to a *-representation if \( A \) is a strongly amenable C*-algebra. It was shown in [3] and [4] that if \( \pi \) is a cyclic representation of \( A \) on a separable Hilbert space, then there exist a one-to-one selfadjoint densely defined unbounded operator \( U \) on \( H \) and a *-representation \( \theta \) of \( A \) on \( H \) such that

\[
U\pi(a)x = \theta(a)Ux
\]

for all \( a \in A \) and \( x \) in the domain of \( U \).

In this paper, we use a result of Pisier [12] and ideas of Christensen [7] and Ringrose [14] to prove that if \( \pi: A \rightarrow B(H) \) is a cyclic representation, then there exist a *-representation \( \theta: A \rightarrow B(H) \) and a one-to-one bounded positive operator \( P \) on \( H \) such that \( P\theta(a) = \pi(a)P \), for all \( a \in A \). We include applications to two related problems: Is every generalized derivation (in the sense of [1]) of \( A \) inner? Is every invariant operator range for \( A \) the range of an operator in the commutant of \( A \)?

2. The main results. The key to our results is the following theorem of Pisier [12, Corollary 2.3].

Theorem 2.1. If \( u: A \rightarrow B \) is a bounded linear map between C*-algebras, then for all \( a_1, a_2, \ldots, a_n \) in \( A \) we have

\[
\left\| \sum_{i=1}^{n} u(a_i)^*u(a_i) + u(a_i)u(a_i)^* \right\| \leq 6\|u\|^2 \left\| \sum_{i=1}^{n} a_i^*a_i + a_ia_i^* \right\|.
\]

Received by the editors February 29, 1980.

AMS (MOS) subject classifications (1970). Primary 46L05.

Key words and phrases. The similarity problem, derivations, invariant operator range, Pisier's inequality.

This research was partially supported by NSF Grant MCS 77-01850.

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0002-9939/81/0000-0115/S02.50

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If \( u \) is a homomorphism, then, as in [7, Theorem 4.1], the above inequality can take a different form.

**Corollary 2.2.** Let \( \pi: A \to B(H) \) be a continuous representation. Then for \( a_1, a_2, \ldots, a_n \in A \) we have

\[
\left\| \sum_{i=1}^n \pi(a_i)^* \pi(a_i) \right\| < 12\|\pi\|^4 \left\| \sum_{i=1}^n a_i^* a_i \right\|.
\]

**Proof.** By [2, Theorem 1] we can extend \( \pi \) to \( \pi_0: A^{**} \to B(H) \), \( \pi_0 \) a homomorphism with the same norm as \( \pi \). Let \( a_i = v_i h_i \) be the polar decomposition of \( a_i \), \( h_i^2 = a_i^* a_i \) and \( v_i \) a partial isometry in \( A^{**} \). Then

\[
\left\| \sum_{i=1}^n \pi(a_i)^* \pi(a_i) \right\| = \left\| \sum \pi(h_i)^* \pi_0(v_i) \pi_0(v_i)^* \pi(h_i) \right\|
\]

\[
< \left\| \sum \|\pi\|^2 \pi(h_i)^* \pi(h_i) \right\|
\]

\[
< \|\pi\|^2 \left\| \sum \pi(h_i)^* \pi(h_i) + \pi(h_i) \pi(h_i)^* \right\|
\]

\[
< 6\|\pi\|^4 \left\| \sum h_i^* h_i + h_i h_i^* \right\|
\]

\[
= 12\|\pi\|^4 \left\| \sum h_i^2 \right\| = 12\|\pi\|^4 \left\| \sum a_i^* a_i \right\|.
\]

As in [14, p. 303], [6, p. 239] or [7], the following theorem follows from the inequality in Corollary 2.2.

**Theorem 2.3.** Let \( \pi: A \to B(H) \) be a continuous representation and let \( g \) be a state on \( B(H) \). Then there exists a state \( f \) on \( A \) such that

\[
g(\pi(a)^* \pi(a)) < 12\|\pi\|^4 f(a^* a).
\]

In the following theorem, \( \pi_f \) is the GNS representation constructed from \( f \).

**Theorem 2.4.** Let \( \pi: A \to B(H) \) be a continuous representation, \( x_0 \in H, \|x_0\| = 1 \). Then there exist a state \( f \) on \( A \) and a bounded operator \( S: H_f \to H, \|S\| < \sqrt{12} \|\pi\|^2, \) such that \( S\pi_f(a) = \pi(a)S \) for all \( a \in A \) and \( \text{Range } S \supseteq \pi(A)x_0 \).

**Proof.** Use Theorem 2.3 with \( g(b) = (bx_0, x_0) \) for all \( b \in B(H) \). Then

\[
\|\pi(a)x_0\| < \sqrt{12} \|\pi\|^2 f(a^* a)^{1/2}.
\]

Let \( K_f = \{ a \in A : f(a^* a) = 0 \} \), and define \( S: A / K_f \to H \) by \( S(a + K_f) = \pi(a)x_0 \). Then this determines \( S: H_f \to H \) with \( \|S\| < \sqrt{12} \|\pi\|^2 \). For any \( a, b \in A \) we have \( S\pi_f(a)(b + K_f) = \pi(a)\pi(b)x_0 = \pi(a)S(b + K_f) \). So \( S\pi_f(a) = \pi(a)S \).

**Corollary 2.5.** Let \( \pi: A \to B(H) \) be a continuous representation with a cyclic vector. Then there exist a \(*\)-representation \( \theta: A \to B(H) \) and a bounded, one-to-one, positive operator \( P \) on \( H \) such that \( P\theta(a) = \pi(a)P \) for all \( a \in A \) and \( \|P\| < \sqrt{12} \|\pi\|^2 \).
Proof. Let $x_0 \in H$ be a cyclic unit vector for $\pi$. Then by Theorem 2.4 there is an operator $S: H_f \to H$, $\|S\| \leq \sqrt{12} \|\pi\|^2$, such that $S\pi_f(a) = \pi(a)S$ and $\text{Range } S \supseteq \pi(A)x_0$. Then $\ker(S)^\perp$ is a reducing subspace for $\pi_f(A)$. Let $\rho(a) = \pi_f(a)\ker(S)^\perp$, $T = S\ker(S)^\perp$. Then $T\rho(a) = \pi(a)T$, and $T$ is one-to-one with dense range. Let $T^* = WP$ be the polar decomposition of $T^*$; $W$ is unitary and $P$ is positive one-to-one with $\|P\| \leq \sqrt{12} \|\pi\|^2$. Then $T = PW^*$ and $PW^*\rho(a) = \pi(a)PW^*$. Let $\theta(a) = W^*\rho(a)W$. Then $\theta: A \to B(H)$ is a $*$-representation and $P\theta(a) = \pi(a)P$.

3. Applications. Let $A$ be a $C^*$-algebra, $\theta: A \to B(H)$ a $*$-representation. Let $D: A \to B(H)$ be a linear map satisfying $D(ab) = \theta(a)D(b) + D(a)\theta(b)$. The map $D$ has been called a $\theta$-derivation [1]. We cannot show directly that $D$ is inner, but we can show, assuming that $\theta$ has a cyclic vector, that there is a closed densely defined operator $h$ such that $D(a) = h\theta(a) - \theta(a)h$ on the domain of $h$. It then follows from [7, Corollary 5.4] and [6, Proposition 2.1], that there is a bounded operator $t$ on $H$ with $D(a) = \theta(a)t - \theta(a)t$ for $a \in A$. The situation is then the same as for derivations of $C^*$-algebras into a containing $B(H)$ [7, Corollary 5.4].

We construct the unbounded operator $h$ as follows. Let $\pi: A \to B(H \oplus H)$ be defined by

$$
\pi(a) = \begin{pmatrix}
\theta(a) & D(a) \\
0 & \theta(a)
\end{pmatrix}
$$

It is then easily seen that $\pi$ is a homomorphism. It follows from [13] that $D$ is automatically continuous, so that $\pi$ is continuous. Let $y_0$ be a cyclic unit vector for $\pi(A)$ and apply Theorem 2.4 to $\pi$ and $x_0 = (y_0, y_0)$. There thus exists a bounded operator $S: H_f \to H \oplus H$ such that $S\pi_f(a) = \pi(a)S$ and $\text{Range } S \supseteq \pi(A)x_0 = \{(D(a)y_0, \theta(a)y_0): a \in A\}$. Then since $\pi_f$ is a $*$-representation, we also have $\pi_f(a)S^* = S^*\pi(a^*)^*$ and $SS^*\pi(a^*) = \pi(a)SS^*$. Writing $SS^*$ as a 2-by-2 operator matrix, this becomes

$$
\begin{pmatrix}
P & Q \\
R & T
\end{pmatrix}
\begin{pmatrix}
\theta(a) & 0 \\
D(a^*) & \theta(a)
\end{pmatrix}
= \begin{pmatrix}
\theta(a) & D(a) \\
0 & \theta(a)
\end{pmatrix}
\begin{pmatrix}
P & Q \\
R & T
\end{pmatrix}.
$$

The $(2, 2)$-entry of this equation yields that $T\theta(a) = \theta(a)T$ for all $a \in A$. The $(1, 2)$-entry says that

$$
D(a)T = Q\theta(a) - \theta(a)Q.
$$

If $Tx = 0$ for $x \in H$, then

$$
\begin{pmatrix}
P & Q \\
R & T
\end{pmatrix}\begin{pmatrix}
0 \\
x
\end{pmatrix} = (Tx, x) = 0,
$$

so $0 \oplus \ker T \subseteq \ker(SS^*) = (\text{Range } S)^\perp$. So if $Tx = 0$, then

$$
0 = \begin{pmatrix}
0 \\
x
\end{pmatrix}, \begin{pmatrix}
D(a)y_0 \\
\theta(a)y_0
\end{pmatrix} = (x, \theta(a)y_0)
$$

for all $a \in A$. So since $y_0$ is cyclic for $\theta$ it follows that $x = 0$, and $T$ is a positive, one-to-one, operator. By replacing $Q$ by a scalar translate, we can assume that $Q$ is invertible. Since $T \in \theta(A)'$ we have that $D(a) = QT^{-1}\theta(a) - \theta(a)QT^{-1}$ on the
range of $T$. Since $Q$ is invertible it is easily seen that $QT^{-1}$ is closed. The following theorem then follows from this, [7, Corollary 5.4] and [6, Proposition 2.1].

**Theorem 3.1.** Let $\theta: A \to B(H)$ be a cyclic $*$-representation of the $C^*$-algebra $A$. Then any $\theta$-derivation $D: A \to B(H)$ is inner.

We now consider a different problem. Let $A$ be a $C^*$-subalgebra of $B(H)$, and let $T \in B(H)$ be such that Range $T$ is an invariant linear manifold for $A$. The invariant operator range problem asks if there is a bounded operator $T'$ in the commutant of $A$ such that Range $T = \text{Range } T'$. To my knowledge, this problem was first raised by Dixmier at the 1967 Baton Rouge $C^*$-algebra conference. It was noted by Foias that the invariant operator range problem is true if every continuous representation of a $C^*$-algebra is similar to a $*$-representation [10]. Our next theorem proves that, conversely, if every invariant operator range for a $C^*$-algebra comes from an operator in the commutant, then every cyclic representation is similar to a $*$-representation.

**Theorem 3.2.** Let $\pi: A \to B(H)$ be a continuous representation with a cyclic vector. If every invariant operator range for a $C^*$-algebra comes from an operator in the commutant, then $\pi$ is similar to a $*$-representation.

**Proof.** By Corollary 2.5 we know that there is a $*$-representation $\theta: A \to B(H)$ and a bounded, positive, one-to-one operator $P$ on $H$ such that $P\theta(a) = \pi(a)P$. Then $\theta(a)P = P\pi(a^*)^*$, so $\theta(A)$ leaves the range of $P$ invariant. By assumption, there is then a bounded operator $R \in \theta(A)'$ such that Range $R = \text{Range } P$. Since $R$ and $|R^*|$ have the same range, we may assume that $R$ is positive and one-to-one. There then exist linear transformations $L_1$ and $L_2$ such that $Rx = PL_1x$, $Px = RL_2x$ for all $x \in H$, and an application of the closed graph theorem shows that $L_1$ and $L_2$ are bounded. Since $P$ and $R$ are one-to-one it follows that $L_1$ and $L_2$ are inverses. Then $\theta(a)L_2 = L_2\pi(a^*)^*$, and $R\theta(a)L_2 = RL_2\pi(a^*)^*$, so $\theta(a)L_2 = L_2\pi(a^*)^*$, or $\pi(a) = L_2^*\theta(a)(L_2^*)^{-1}$, where $L_2$ is a bounded operator with a bounded inverse.

The positive results on the similarity problem that were obtained in §2 do not seem to be sufficient to prove the invariant operator range problem, but the following partial results can be proved.

**Theorem 3.3.** Let $T \in B(H)$ be such that $T(H)$ is invariant for a $C^*$-subalgebra $A$ of $B(H)$. Then for any $x_0 \in H$, there is an operator $P \in A'$ such that $Tx_0 \in P(H) \subseteq T(H)$.

**Proof.** It clearly suffices to assume that $T > 0$ and $T$ is one-to-one, for otherwise we may cut $A$ down to the closure of $T(H)$. We proceed as in [10, p. 890]. For $a \in A$ and $x \in H$, there is a unique $y \in H$ such that $T x = Ty$. Let $\pi(a)x = y$. Several applications of the closed graph theorem show that $\pi: A \to B(H)$ is a continuous homomorphism, with $aT = T\pi(a)$ for all $a \in A$. By Theorem 2.4, there is a bounded operator $S$ such that $S\theta(a) = \pi(a)S$, for $\theta$ a $*$-representation of $A$, and $\pi(A)x_0 \subseteq \text{Range } S$. By restricting $\theta$ to $\ker(S)^\perp$, we may assume that

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$S$ is one-to-one. From the four equations

\begin{align*}
aT &= T\pi(a), & S\theta(a) &= \pi(a)S, \\
Ta &= \pi(a^*)^*T, & \theta(a)S^* &= S^*\pi(a^*)^*
\end{align*}

we obtain that

\begin{align*}
TSS^*Ta &= TSS^*\pi(a^*)^*T = TS\theta(a)S^*T \\
&= T\pi(a)SS^*T = aTSS^*T.
\end{align*}

So $TSS^*T$ is in the commutant of $A$. Let $P$ be the positive square root of $TSS^*T$. Then $P \in A'$, and $
\text{Range } P = \text{Range } TS \supset \{ T\pi(a)x_0 : a \in A \}$ so $ATx_0 \subseteq \text{Range } P \subseteq \text{Range } T$.

**Corollary 3.4.** Let $T \in B(H)$ be such that $T(H)$ is invariant for a $C^*$-subalgebra $A$ of $B(H)$. Then $T(H)$ is also invariant for the weak closure of $A$.

**Proof.** Let $x \in T(H)$ and choose, by Theorem 3.3, an operator $P \in A'$ such that $x \in P(H) \subset T(H)$. Then for all $a \in A''$, $ax \in aP(H) \subseteq P(H) \subseteq T(H)$, so $T(H)$ is invariant for $A''$.

We close with the following theorem, which is probably known to many people.

**Theorem 3.5.** Let $A$ be a nuclear $C^*$-algebra. Then any continuous representation $\pi: A \to B(H)$ is similar to a $\pi^*$-representation.

**Proof.** By [9], $A^{**}$ has an ultraweakly dense $C^*$-subalgebra $B$ which is the norm-closed linear span of an amenable group $G$ of unitaries. By [2, Theorem 1] we can extend $\pi$ to $\pi_0: A^{**} \to B(H)$ with $\pi_0$ an ultraweak to ultraweak continuous homomorphism with the same norm as $\pi$. By an old result of Dixmier [8], there is a bounded invertible operator $S$ such that $\theta(u) = S^{-1}\pi_0(u)S$ is a continuous unitary representation of $G$. Then $SS^*\pi_0(u^*)^* = \pi_0(u)SS^*$ for all $u \in G$. It follows that $SS^*\pi_0(a^*)^* = \pi_0(a)SS^*$ for all $a \in A^{**}$. Let $P$ be the positive square root of $SS^*$. Define a map $\rho: A \to B(H)$ by $\rho(a) = P^{-1}\pi(a)P$. Since $P\pi(a^*)^*P^{-1} = P^{-1}\pi(a)P$, it is immediate that $\rho$ is a $\pi^*$-representation of $A$.

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