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THE SIMILARITY PROBLEM FOR REPRESENTATIONS OF C^*-ALGEBRAS

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ABSTRACT. Let $\pi: A \to B(H)$ be a bounded homomorphism of a C^*-algebra into the bounded operators on a Hilbert space. We prove that, if π is cyclic, there is a $*$-representation $\theta: A \to B(H)$ and a bounded one-to-one positive operator P such that $P\theta(a) = \pi(a)P$. We include applications to θ-derivations and invariant operator ranges for operator algebras.

1. Introduction. Let $\pi: A \to B(H)$ be a continuous representation of a C^*-algebra A into the bounded operators on a Hilbert space H. In this paper, we are concerned with whether or not there exist a $*$-representation $\theta: A \to B(H_\theta)$ and a bounded invertible operator $S: H_\theta \to H$ such that $S\theta(a) = \pi(a)S$ for all $a \in A$. That is, must π be similar to a $*$-representation? In the context of C^*-algebras, the question was first raised by Kadison in [**11**]. It was shown in [**5**] that π is similar to a $*$-representation if A is a strongly amenable C^*-algebra. It was shown in [**3**] and [**4**] that if π is a cyclic representation of A on a separable Hilbert space, then there exist a one-to-one selfadjoint densely defined unbounded operator U on H and a $*$-representation θ of A on H such that

$$U\pi(a)x = \theta(a)Ux$$

for all $a \in A$ and x in the domain of U.

In this paper, we use a result of Pisier [**12**] and ideas of Christensen [**7**] and Ringrose [**14**] to prove that if $\pi: A \to B(H)$ is a cyclic representation, then there exist a $*$-representation $\theta: A \to B(H)$ and a one-to-one bounded positive operator P on H such that $P\theta(a) = \pi(a)P$, for all $a \in A$. We include applications to two related problems: Is every generalized derivation (in the sense of [**1**]) of A inner? Is every invariant operator range for A the range of an operator in the commutant of A?

2. The main results. The key to our results is the following theorem of Pisier [**12**, Corollary 2.3].

THEOREM 2.1. *If $u: A \to B$ is a bounded linear map between C^*-algebras, then for all $a_1, a_2, \ldots, a_n$ in A we have*

$$\left\| \sum_{i=1}^{n} u(a_i)^* u(a_i) + u(a_i)u(a_i)^* \right\| \leqslant 6\|u\|^2 \left\| \sum_{i=1}^{n} a_i^* a_i + a_i a_i^* \right\|.$$

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If u is a homomorphism, then, as in [7, Theorem 4.1], the above inequality can take a different form.

COROLLARY 2.2. *Let π: $A \to B(H)$ be a continuous representation. Then for a_1, $a_2, \ldots, a_n \in A$ we have*

$$\left\| \sum_{i=1}^{n} \pi(a_i)^* \pi(a_i) \right\| \leq 12 \|\pi\|^4 \left\| \sum_{i=1}^{n} a_i^* a_i \right\|.$$

PROOF. By [2, Theorem 1] we can extend π to π_0: $A^{**} \to B(H)$, π_0 a homomorphism with the same norm as π. Let $a_i = v_i h_i$ be the polar decomposition of a_i, $h_i^2 = a_i^* a_i$ and v_i a partial isometry in A^{**}. Then

$$\begin{aligned} \left\| \sum_{1}^{n} \pi(a_i)^* \pi(a_i) \right\| &= \left\| \sum \pi(h_i)^* \pi_0(v_i)^* \pi_0(v_i) \pi(h_i) \right\| \\ &\leq \left\| \sum \|\pi\|^2 \pi(h_i)^* \pi(h_i) \right\| \\ &\leq \|\pi\|^2 \left\| \sum \pi(h_i)^* \pi(h_i) + \pi(h_i) \pi(h_i)^* \right\| \\ &\leq 6 \|\pi\|^4 \left\| \sum h_i^* h_i + h_i h_i^* \right\| \\ &= 12 \|\pi\|^4 \left\| \sum h_i^2 \right\| = 12 \|\pi\|^4 \left\| \sum_{1}^{n} a_i^* a_i \right\|. \end{aligned}$$

As in [**14**, p. 303], [**6**, p. 239] or [**7**], the following theorem follows from the inequality in Corollary 2.2.

THEOREM 2.3. *Let π: $A \to B(H)$ be a continuous representation and let g be a state on $B(H)$. Then there exists a state f on A such that*

$$g(\pi(a)^* \pi(a)) \leq 12 \|\pi\|^4 f(a^* a).$$

In the following theorem, π_f is the GNS representation constructed from f.

THEOREM 2.4. *Let π: $A \to B(H)$ be a continuous representation, $x_0 \in H$, $\|x_0\| = 1$. Then there exist a state f on A and a bounded operator S: $H_f \to H$, $\|S\| \leq \sqrt{12}\, \|\pi\|^2$, such that $S\pi_f(a) = \pi(a)S$ for all $a \in A$ and* (Range S) $\supseteq \pi(A)x_0$.

PROOF. Use Theorem 2.3 with $g(b) = (bx_0, x_0)$ for all $b \in B(H)$. Then

$$\|\pi(a)x_0\| \leq \sqrt{12}\, \|\pi\|^2 f(a^* a)^{1/2}.$$

Let $K_f = \{a \in A : f(a^* a) = 0\}$, and define S: $A/K_f \to H$ by $S(a + K_f) = \pi(a)x_0$. Then this determines S: $H_f \to H$ with $\|S\| \leq \sqrt{12}\, \|\pi\|^2$. For any $a, b \in A$ we have $S\pi_f(a)(b + K_f) = \pi(a)\pi(b)x_0 = \pi(a)S(b + K_f)$. So $S\pi_f(a) = \pi(a)S$.

COROLLARY 2.5. *Let π: $A \to B(H)$ be a continuous representation with a cyclic vector. Then there exist a *-representation θ: $A \to B(H)$ and a bounded, one-to-one, positive operator P on H such that $P\theta(a) = \pi(a)P$ for all $a \in A$ and $\|P\| \leq \sqrt{12}\, \|\pi\|^2$.*

PROOF. Let $x_0 \in H$ be a cyclic unit vector for π. Then by Theorem 2.4 there is an operator S: $H_f \to H$, $\|S\| < \sqrt{12}\, \|\pi\|^2$, such that $S\pi_f(a) = \pi(a)S$ and Range $S \supseteq \pi(A)x_0$. Then $\ker(S)^\perp$ is a reducing subspace for $\pi_f(A)$. Let $\rho(a) = \pi_f(a)|\ker(S)^\perp$, $T = S|\ker(S)^\perp$. Then $T\rho(a) = \pi(a)T$, and T is one-to-one with dense range. Let $T^* = WP$ be the polar decomposition of T^*; W is unitary and P is positive one-to-one with $\|P\| < \sqrt{12}\, \|\pi\|^2$. Then $T = PW^*$ and $PW^*\rho(a) = \pi(a)PW^*$. Let $\theta(a) = W^*\rho(a)W$. Then θ: $A \to B(H)$ is a $*$-representation and $P\theta(a) = \pi(a)P$.

3. Applications. Let A be a C^*-algebra, θ: $A \to B(H)$ a $*$-representation. Let D: $A \to B(H)$ be a linear map satisfying $D(ab) = \theta(a)D(b) + D(a)\theta(b)$. The map D has been called a θ-derivation [**1**]. We cannot show directly that D is inner, but we can show, assuming that θ has a cyclic vector, that there is a closed densely defined operator h such that $D(a) = h\theta(a) - \theta(a)h$ on the domain of h. It then follows from [**7**, Corollary 5.4] and [**6**, Proposition 2.1], that there is a bounded operator t on H with $D(a) = \theta(a)t - \theta(a)t$ for $a \in A$. The situation is then the same as for derivations of C^*-algebras into a containing $B(H)$ [**7**, Corollary 5.4].

We construct the unbounded operator h as follows. Let π: $A \to B(H \oplus H)$ be defined by

$$\pi(a) = \begin{pmatrix} \theta(a) & D(a) \\ 0 & \theta(a) \end{pmatrix}.$$

It is then easily seen that π is a homomorphism. It follows from [**13**] that D is automatically continuous, so that π is continuous. Let y_0 be a cyclic unit vector for $\theta(A)$ and apply Theorem 2.4 to π and $x_0 = (0, y_0)$. There thus exists a bounded operator S: $H_f \to H \oplus H$ such that $S\pi_f(a) = \pi(a)S$ and Range $S \supseteq \pi(A)x_0 = \{(D(a)y_0, \theta(a)y_0) : a \in A\}$. Then since π_f is a $*$-representation, we also have $\pi_f(a)S^* = S^*\pi(a^*)^*$ and $SS^*\pi(a^*)^* = \pi(a)SS^*$. Writing SS^* as a 2-by-2 operator matrix, this becomes

$$\begin{pmatrix} P & Q \\ R & T \end{pmatrix}\begin{pmatrix} \theta(a) & 0 \\ D(a^*)^* & \theta(a) \end{pmatrix} = \begin{pmatrix} \theta(a) & D(a) \\ 0 & \theta(a) \end{pmatrix}\begin{pmatrix} P & Q \\ R & T \end{pmatrix}.$$

The $(2, 2)$-entry of this equation yields that $T\theta(a) = \theta(a)T$ for all $a \in A$. The $(1, 2)$-entry says that

$$D(a)T = Q\theta(a) - \theta(a)Q.$$

If $Tx = 0$ for $x \in H$, then

$$\left(\begin{pmatrix} P & Q \\ P & T \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}\right) = (Tx, x) = 0,$$

so $0 \oplus \ker T \subseteq \ker(SS^*) = (\text{Range } S)^\perp$. So if $Tx = 0$, then

$$0 = \left(\begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} D(a)y_0 \\ \theta(a)y_0 \end{pmatrix}\right) = (x, \theta(a)y_0)$$

for all $a \in A$. So since y_0 is cyclic for θ it follows that $x = 0$, and T is a positive, one-to-one, operator. By replacing Q by a scalar translate, we can assume that Q is invertible. Since $T \in \theta(A)'$ we have that $D(a) = QT^{-1}\theta(a) - \theta(a)QT^{-1}$ on the range of T. Since Q is invertible it is easily seen that QT^{-1} is closed. The following theorem then follows from this, [7, Corollary 5.4] and [6, Proposition 2.1].

THEOREM 3.1. *Let θ: $A \to B(H)$ be a cyclic *-representation of the C^*-algebra A. Then any θ-derivation D: $A \to B(H)$ is inner.*

We now consider a different problem. Let A be a C^*-subalgebra of $B(H)$, and let $T \in B(H)$ be such that Range T is an invariant linear manifold for A. The invariant operator range problem asks if there is a bounded operator T' in the commutant of A such that Range T = Range T'. To my knowledge, this problem was first raised by Dixmier at the 1967 Baton Rouge C^*-algebra conference. It was noted by Foias that the invariant operator range problem is true if every continuous representation of a C^*-algebra is similar to a *-representation [10]. Our next theorem proves that, conversely, if every invariant operator range for a C^*-algebra comes from an operator in the commutant, then every cyclic representation is similar to a *-representation.

THEOREM 3.2. *Let π: $A \to B(H)$ be a continuous representation with a cyclic vector. If every invariant operator range for a C^*-algebra comes from an operator in the commutant, then π is similar to a *-representation.*

PROOF. By Corollary 2.5 we know that there is a *-representation θ: $A \to B(H)$ and a bounded, positive, one-to-one operator P on H such that $P\theta(a) = \pi(a)P$. Then $\theta(a)P = P\pi(a^*)^*$, so $\theta(A)$ leaves the range of P invariant. By assumption, there is then a bounded operator $R \in \theta(A)'$ such that Range R = Range P. Since R and $|R^*|$ have the same range, we may assume that R is positive and one-to-one. There then exist linear tranformations L_1 and L_2 such that $Rx = PL_1x$, $Px = RL_2x$ for all $x \in H$, and an application of the closed graph theorem shows that L_1 and L_2 are bounded. Since P and R are one-to-one it follows that L_1 and L_2 are inverses. Then $\theta(a)RL_2 = RL_2\pi(a^*)^*$, and $R\theta(a)L_2 = RL_2\pi(a^*)^*$, so $\theta(a)L_2 = L_2\pi(a^*)^*$, or $\pi(a) = L_2^*\theta(a)(L_2^*)^{-1}$, where L_2 is a bounded operator with a bounded inverse.

The positive results on the similarity problem that were obtained in §2 do not seem to be sufficient to prove the invariant operator range problem, but the following partial results can be proved.

THEOREM 3.3. *Let $T \in B(H)$ be such that $T(H)$ is invariant for a C^*-subalgebra A of $B(H)$. Then for any $x_0 \in H$, there is an operator $P \in A'$ such that*

$$Tx_0 \in P(H) \subseteq T(H).$$

PROOF. It clearly suffices to assume that $T \geqslant 0$ and T is one-to-one, for otherwise we may cut A down to the closure of $T(H)$. We proceed as in [10, p. 890]. For $a \in A$ and $x \in H$, there is a unique $y \in H$ such that a $Tx = Ty$. Let $\pi(a)x = y$. Several applications of the closed graph theorem show that π: $A \to B(H)$ is a continuous homomorphism, with $aT = T\pi(a)$ for all $a \in A$. By Theorem 2.4, there is a bounded operator S such that $S\theta(a) = \pi(a)S$, for θ a *-representation of A, and $\pi(A)x_0 \subseteq$ Range S. By restricting θ to $\ker(S)^{\perp}$, we may assume that

S is one-to-one. From the four equations

$$aT = T\pi(a), \qquad S\theta(a) = \pi(a)S,$$
$$Ta = \pi(a^*)^*T, \qquad \theta(a)S^* = S^*\pi(a^*)^*$$

we obtain that

$$\begin{aligned} TSS^*Ta &= TSS^*\pi(a^*)^*T = TS\theta(a)S^*T \\ &= T\pi(a)SS^*T = aTSS^*T. \end{aligned}$$

So TSS^*T is in the commutant of A. Let P be the positive square root of TSS^*T. Then $P \in A'$, and Range P = Range $TS \supseteq \{T\pi(a)x_0\colon a \in A\}$ so $ATx_0 \subseteq$ Range $P \subseteq$ Range T.

COROLLARY 3.4. *Let $T \in B(H)$ be such that $T(H)$ is invariant for a C^*-subalgebra A of $B(H)$. Then $T(H)$ is also invariant for the weak closure of A.*

PROOF. Let $x \in T(H)$ and choose, by Theorem 3.3, an operator $P \in A'$ such that $x \in P(H) \subset T(H)$. Then for all $a \in A''$, $ax \in aP(H) \subseteq P(H) \subseteq T(H)$, so $T(H)$ is invariant for A''.

We close with the following theorem, which is probably known to many people.

THEOREM 3.5. *Let A be a nuclear C^*-algebra. Then any continuous representation $\pi\colon A \to B(H)$ is similar to a $*$-representation.*

PROOF. By **[9]**, A^{**} has an ultraweakly dense C^*-subalgebra B which is the norm-closed linear span of an amenable group G of unitaries. By **[2**, Theorem 1**]** we can extend π to $\pi_0\colon A^{**} \to B(H)$ with π_0 an ultraweak to ultraweak continuous homomorphism with the same norm as π. By an old result of Dixmier **[8]**, there is a bounded invertible operator S such that $\theta(u) = S^{-1}\pi_0(u)S$ is a continuous unitary representation of G. Then $SS^*\pi_0(u^*)^* = \pi_0(u)SS^*$ for all $u \in G$. It follows that $SS^*\pi_0(a^*)^* = \pi_0(a)SS^*$ for all a in A^{**}. Let P be the positive square root of SS^*. Define a map $\rho\colon A \to B(H)$ by $\rho(a) = P^{-1}\pi(a)P$. Since $P\pi(a^*)^*P^{-1} = P^{-1}\pi(a)P$, it is immediate that ρ is a $*$-representation of A.

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