NEW SUPPORT POINTS OF $S$ AND EXTREME POINTS OF $KC_S$

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ABSTRACT. Let $S$ be the usual class of univalent analytic functions $f$ on $\{z||z|<1\}$ normalized by $f(z) = z + a_2z^2 + \cdots$. We prove that the functions

$$f_{w,v}(z) = \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2}, \quad |x| = |y| = 1, x \neq y,$$

which are support points of $C$, the subclass of $S$ of close-to-convex functions, and extreme points of $KC_S$ whenever $0 < |\arg(-x/y)| < \pi/4$. We observe that the known bound of $\pi/4$ for the acute angle between the omitted arc of a support point of $S$ and the radius vector is achieved by the functions $f_{w,v}$ with $|\arg(-x/y)| = \pi/4$.

Introduction. Let $\mathfrak{A}$ be the set of analytic functions on the open unit disk. With the usual topology of uniform convergence on compacta $\mathfrak{A}$ is a locally convex linear topological space. Suppose $\mathbb{B} \subset \mathfrak{A}$. A function $b$ in $\mathbb{B}$ is called a support point of $\mathbb{B}$ if $b$ maximizes $\text{Re } J$ over $\mathbb{B}$ for some continuous linear functional $J$ on $\mathfrak{A}$ such that $\text{Re } J$ is not constant on $\mathfrak{A}$. Let $\mathfrak{K} \mathfrak{B}$ denote the closed convex hull of $\mathbb{B}$. A function $b$ in $\mathfrak{K} \mathfrak{B}$ is called an extreme point of $\mathfrak{K} \mathfrak{B}$ if $b = t_1b_1 + (1 - t)b_2$ implies $b = b_1 = b_2$ whenever $0 < t < 1$ and $b_1, b_2 \in \mathfrak{K} \mathfrak{B}$.

Let $S$ be the usual class of univalent functions $f$ in $\mathfrak{A}$ normalized by $f(z) = z + a_2z^2 + \cdots$. A. Pfluger [10] and L. Brickman and D. R. Wilken [3] have shown that if $f$ is a support point of $S$, then $f$ maps the open unit disk to the complement of an analytic arc $\Gamma$, which tends to $\infty$ with increasing modulus. Furthermore, $\Gamma$ satisfies the $\pi/4$-property, i.e., if $\Gamma$ is oriented so that $\Gamma$ is (positively) traversed from the finite tip to $\infty$, then the angle between the oriented tangent vector to $\Gamma$ and the radius vector to $\Gamma$ at any point is less than or equal to $\pi/4$, with strict inequality at each point of $\Gamma$ except possibly at the finite tip.

In an early paper [1] L. Brickman proved that if $f$ in $S$ is an extreme point of $KC_S$, then $f$ maps the open unit disk to the complement of an arc which tends to $\infty$ with increasing modulus. Later W. E. Kirwan and R. W. Pell [9] improved Brickman's result. A special case of their result states that if $f$ in $S$ is an extreme point of $KC_S$ and if the omitted arc of $f$ is smooth, then the omitted arc of $f$ satisfies the $\pi/4$-property, albeit, not necessarily with strict inequality.

Since $S$ and $KC_S$ are compact a lemma in Dunford and Schwartz [5, p. 440] implies that if $f$ is a support point of $S$, then $f \in S$. The following lemma shows that in certain cases we can identify support points of $S$ as extreme points of $KC_S$.

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Lemma. Let $J$ be a continuous linear functional on $\mathcal{S}$ such that $\text{Re} J$ is nonconstant on $\mathcal{S}$. If there exist at most two support points of $\mathcal{S}$ which maximize $\text{Re} J$ over $\mathcal{S}$, then each such support point of $\mathcal{S}$ is an extreme point of $\mathcal{K}\mathcal{S}$.

It is well known that the Koebe functions $k_x(z) = z/(1 - xz)^2$, $|x| = 1$, uniquely maximize $\text{Re} J_x$ over $\mathcal{S}$, where $J_x g = g''(0)$, $|x| = 1$. Thus, the Koebe functions $k_x, |x| = 1$, are both support points of $\mathcal{S}$ and extreme points of $\mathcal{K}\mathcal{S}$. Until recently, no other support points of $\mathcal{S}$ or extreme points of $\mathcal{K}\mathcal{S}$ were explicitly known. However, J. Brown [4] has determined the support points of $\mathcal{S}$ which maximize $\text{Re} J$ over $\mathcal{S}$, where $J g = g(z_0), 0 < |z_0| < 1$, and that each such support point of $\mathcal{S}$ is an extreme point of $\mathcal{K}\mathcal{S}$.

The class $\mathcal{C}$. Let $\mathcal{C}$ be the subclass of $\mathcal{S}$ of close-to-convex functions. In [2] L. Brickman, T. H. MacGregor, and D. R. Wilken showed that the extreme points of $\mathcal{K}\mathcal{C}$ are the functions

$$f_{x,y}(z) = \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2}, \quad |x| = |y| = 1, \ x \neq y. \quad (1)$$

Later E. Grassman, W. Hengartner, and G. Schober [7] proved that each support point of $\mathcal{C}$ is a function of the form (1). In [8] D. R. Wilken and R. Hornblower showed that each extreme point of $\mathcal{K}\mathcal{C}$ is a support point of $\mathcal{C}$.

A natural question arises as to whether the functions (1) are support points of $\mathcal{S}$ or extreme points of $\mathcal{K}\mathcal{S}$. Each function $f_{x,y}$ in (1) maps the open unit disk to the complement of a half-line. Let $\Gamma_{x,y}$, the omitted half-line of $f_{x,y}$, be oriented so that $\Gamma_{x,y}$ is traversed from $P_{x,y}$, the finite tip of $\Gamma_{x,y}$, to $\infty$. A computation shows that $|\arg(-x/y)|$ is the angle between the tangent vector to $\Gamma_{x,y}$ and the radius vector to $\Gamma_{x,y}$ at $P_{x,y}$. It is easily seen that the angle between the tangent vector to $\Gamma_{x,y}$ and the radius vector to $\Gamma_{x,y}$ decreases (monotonically) to 0 as $\Gamma_{x,y}$ is traversed (monotonically) from $P_{x,y}$ to $\infty$. Thus, if $\pi/4 < |\arg(-x/y)| < \pi$, then $f_{x,y}$ can be neither a support point of $\mathcal{S}$ nor an extreme point of $\mathcal{K}\mathcal{S}$ (because $\Gamma_{x,y}$ fails to satisfy the $\pi/4$-property). If $|\arg(-x/y)| = 0$, i.e., if $-x = y$, then $f_{x,y}$ is the Koebe function $k_x$ and is both a support point of $\mathcal{S}$ and an extreme point of $\mathcal{K}\mathcal{S}$. In the remaining case $-\pi/4 < |\arg(-x/y)| < \pi/4$, $f_{x,y}$ does not violate the $\pi/4$-property. We will show for $0 < |\arg(-x/y)| < \pi/4$ that $f_{x,y}$ is both a support point of $\mathcal{S}$ and an extreme point of $\mathcal{K}\mathcal{S}$.

To prove the main result of this paper, we recall the bound on $|\arg f'(z_0)|$ for $f$ in $\mathcal{S}$ given by G. M. Goluzin [6, p. 115]. Namely, Goluzin showed that if $f \in \mathcal{S}$, then

$$|\arg f'(z_0)| < 4 \arcsin |z_0|, \quad |z_0| < \frac{1}{\sqrt{2}}. \quad (2)$$

We now prove

**Theorem.** Let $f_{x,y}$ be given by (1). If $0 < |\arg(-x/y)| < \pi/4$, then $f_{x,y}$ is both a support point of $\mathcal{S}$ and an extreme point of $\mathcal{K}\mathcal{S}$. 

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PROOF. If we differentiate $f_{x,y}$ and then evaluate at $z_0$, we have

$$f'_{x,y}(z_0) = \frac{1 - xz_0}{(1 - yz_0)^3}.$$  

An easy argument shows for $0 < |z_0| < 1$ that

$$|\arg f'_{x,y}(z_0)| < 4 \arcsin|z_0|$$  

and that equality occurs in (3) if and only if

$$\arg xz_0 = -\arccos|z_0|, \quad \arg yz_0 = \arccos|z_0|$$  

or

$$\arg xz_0 = \arccos|z_0|, \quad \arg yz_0 = -\arccos|z_0|.$$  

If (4) holds, then $\arg f'_{x,y}(z_0) = 4 \arcsin|z_0|$ and if (5) holds, then $\arg f'_{x,y}(z_0) = -4 \arcsin|z_0|$. We note that for each pair $(x,y)$, $|x| = |y| = 1$, $x^2 \neq y^2$, there exists a unique $z_0$, $0 < |z_0| < 1$, such that exactly one of (4) or (5) holds.

Let $0 < |\arg(-x/y)| < \pi/4$ and suppose $z_0$ satisfies (4). Then (4) implies $0 < |z_0| < \sin \pi/8$. Goluzin's bound (2) on $|\arg f'(z_0)|$ implies that the region of variability of $f'(z_0)$ for $f$ in $S$ lies in a closed sector in the closed right half-plane. Together (2)–(4) imply that $f'_{x,y}(z_0)$ lies on the upper edge of the region of variability of $f'(z_0)$ for $f$ in $S$. By rotating the region of variability of $f'(z_0)$ for $f$ in $S$ we can realize a continuous linear functional $J_{x,y}$ whose real part is maximized over $S$ by $f_{x,y}$; namely

$$J_{x,y}g = -e^{i(\pi/2 - 4 \arcsin|z_0|)}g'(z_0).$$  

Similarly, if $0 < |\arg(-x/y)| < \pi/4$ and $z_0$ satisfies (5), then $f_{x,y}$ maximizes $\text{Re } J_{x,y}$ over $S$ where

$$J_{x,y}g = -e^{-i(\pi/2 - 4 \arcsin|z_0|)}g'(z_0).$$  

We will show now that if $0 < |\arg(-x/y)| < \pi/4$, then $\text{Re } J_{x,y}$ is uniquely maximized over $S$ by $f_{x,y}$, and if $|\arg(-x/y)| = \pi/4$, then $\text{Re } J_{x,y}$ is maximized over $S$ (only) by $f_{x,y}$ and $f_{y,x}$. The lemma will then imply if $0 < |\arg(-x/y)| < \pi/4$, then $f_{x,y}$ is an extreme point of $\mathcal{K}^S$.

As in the first part, we can see that if $0 < |z_0| < \sin \pi/8$ and $f^*$ in $S$ maximizes (minimizes) $\arg f'(z_0)$ over $S$, then $f^*$ is a support point of $S$ and, hence, in particular, a slit mapping. Goluzin's argument [6, p. 115], which shows that (2) is sharp, also shows that for $0 < |z_0| < 1/\sqrt{2}$ there exists a unique slit mapping which maximizes (minimizes) $\arg f'(z_0)$ over $S$.

Let $0 < |\arg(-x/y)| < \pi/4$ and let $z_0$ satisfy (4). Since determining the functions which maximize $\text{Re } J_{x,y}$ over $S$ is equivalent to determining the functions which maximize $\arg f'(z_0)$ over $S$, we conclude from the above that $\text{Re } J_{x,y}$ is uniquely maximized over $S$ by $f_{x,y}$. Similarly, if $0 < |\arg(-x/y)| < \pi/4$ and $z_0$ satisfies (5), then $\text{Re } J_{x,y}$ is uniquely maximized over $S$ by $f_{x,y}$.

Let $|\arg(-x/y)| = \pi/4$ and let $z_0$ satisfy (4) or (5). It is easily seen, from (2)–(5) that one of $f_{x,y}$ and $f_{y,x}$ maximizes $\arg f'(z_0)$ over $S$ and the other minimizes $\arg f'(z_0)$ over $S$. Since, in this case, we have $|z_0| = \sin \pi/8$, it follows that
$J_{x,y} = J_{y,x}$. Thus, determining the functions which maximize $\text{Re} J_{x,y}$ over $\mathcal{S}$ is equivalent to determining the functions which maximize or minimize $\arg f'(z_0)$ over $\mathcal{S}$. Consequently, $\text{Re} J_{x,y}$ is maximized over $\mathcal{S}$ (only) by $f_{x,y}$ and $f_{y,x}$.

**Remark.** For the functions $f_{x,y}$ with $|\arg(-x/y)| = \pi/4$, the known bound of $\pi/4$ for the acute angle between the omitted arc of a support point of $\mathcal{S}$ and the radius vector is achieved (at the finite tip).

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**Bibliography**


4. J. E. Brown, *Geometric properties of a class of support points of univalent functions* (manuscript).


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