PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 3, March 1981

# NEW SUPPORT POINTS OF $\mathcal{S}$ AND EXTREME POINTS OF $\mathcal{HS}$

KENT PEARCE

ABSTRACT. Let $\mathcal{S}$ be the usual class of univalent analytic functions $f$ on $\{z||z| < 1\}$ normalized by $f(z) = z + a_2z^2 + \cdots$. We prove that the functions

$$f_{x,y}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, \qquad |x| = |y| = 1, x \neq y,$$

which are support points of $\mathcal{C}$, the subclass of $\mathcal{S}$ of close-to-convex functions, and extreme points of $\mathcal{HC}$, are support points of $\mathcal{S}$ and extreme points of $\mathcal{HS}$ whenever $0 < |\arg(-x/y)| < \pi/4$. We observe that the known bound of $\pi/4$ for the acute angle between the omitted arc of a support point of $\mathcal{S}$ and the radius vector is achieved by the functions $f_{x,y}$ with $|\arg(-x/y)| = \pi/4$.

**Introduction.** Let $\mathcal{A}$ be the set of analytic functions on the open unit disk. With the usual topology of uniform convergence on compacta $\mathcal{A}$ is a locally convex linear topological space. Suppose $\mathcal{B} \subset \mathcal{A}$. A function $b$ in $\mathcal{B}$ is called a support point of $\mathcal{B}$ if $b$ maximizes Re $J$ over $\mathcal{B}$ for some continuous linear functional $J$ on $\mathcal{A}$ such that Re $J$ is not constant on $\mathcal{B}$. Let $\mathcal{HB}$ denote the closed convex hull of $\mathcal{B}$. A function $b$ in $\mathcal{HB}$ is called an extreme point of $\mathcal{HB}$ if $b = tb_1 + (1-t)b_2$ implies $b = b_1 = b_2$ whenever $0 < t < 1$ and $b_1, b_2 \in \mathcal{HB}$.

Let $\mathcal{S}$ be the usual class of univalent functions $f$ in $\mathcal{A}$ normalized by $f(z) = z + a_2z^2 + \cdots$. A. Pfluger [10] and L. Brickman and D. R. Wilken [3] have shown that if $f$ is a support point of $\mathcal{S}$, then $f$ maps the open unit disk to the complement of an analytic arc $\Gamma$, which tends to $\infty$ with increasing modulus. Furthermore, $\Gamma$ satisfies the $\pi/4$-property, i.e., if $\Gamma$ is oriented so that $\Gamma$ is (positively) traversed from the finite tip to $\infty$, then the angle between the oriented tangent vector to $\Gamma$ and the radius vector to $\Gamma$ at any point is less than or equal to $\pi/4$, with strict inequality at each point of $\Gamma$ except possibly at the finite tip.

In an early paper [1] L. Brickman proved that if $f$ in $\mathcal{S}$ is an extreme point of $\mathcal{HS}$, then $f$ maps the open unit disk to the complement of an arc which tends to $\infty$ with increasing modulus. Later W. E. Kirwan and R. W. Pell [9] improved Brickman's result. A special case of their result states that if $f$ in $\mathcal{S}$ is an extreme point of $\mathcal{HS}$ and if the omitted arc of $f$ is smooth, then the omitted arc of $f$ satisfies the $\pi/4$-property, albeit, not necessarily with strict inequality.

Since $\mathcal{S}$ and $\mathcal{HS}$ are compact a lemma in Dunford and Schwartz [5, p. 440] implies that if $f$ is an extreme point of $\mathcal{HS}$, then $f \in \mathcal{S}$. The following lemma shows that in certain cases we can identify support points of $\mathcal{S}$ as extreme points of $\mathcal{HS}$.

Received by the editors February 10, 1980; presented to the Society, August 19, 1980.
1980 *Mathematics Subject Classification*. Primary 30C75; Secondary 30C45.
*Key words and phrases*. Support points, extreme points, univalent functions, close-to-convex functions.

LEMMA. *Let $J$ be a continous linear functional on $\mathcal{A}$ such that* $\operatorname{Re} J$ *is nonconstant on* $\mathcal{S}$. *If there exist at most two support points of* $\mathcal{S}$ *which maximize* $\operatorname{Re} J$ *over* $\mathcal{S}$, *then each such support point of* $\mathcal{S}$ *is an extreme point of* $\mathcal{HS}$.

It is well known that the Koebe functions $k_x(z) = z/(1 - xz)^2$, $|x| = 1$, uniquely maximize $\operatorname{Re} J_x$ over $\mathcal{S}$, where $J_x g = \bar{x} g''(0)$, $|x| = 1$. Thus, the Koebe functions $k_x$, $|x| = 1$, are both support points of $\mathcal{S}$ and extreme points of $\mathcal{HS}$. Until recently, no other support points of $\mathcal{S}$ or extreme points of $\mathcal{HS}$ were explicitly known. However, J. Brown [4] has determined the support points of $\mathcal{S}$ which maximize $\operatorname{Re} J$ over $\mathcal{S}$, where $Jg = g(z_0)$, $0 < |z_0| < 1$, and that each such support point of $\mathcal{S}$ is an extreme point of $\mathcal{HS}$.

**The class** $\mathcal{C}$. Let $\mathcal{C}$ be the subclass of $\mathcal{S}$ of close-to-convex functions. In [2] L. Brickman, T. H. MacGregor, and D. R. Wilken showed that the extreme points of $\mathcal{HC}$ are the functions

$$f_{xy}(z) = \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2}, \qquad |x| = |y| = 1, x \neq y. \tag{1}$$

Later E. Grassman, W. Hengartner, and G. Schober [7] proved that each support point of $\mathcal{C}$ is a function of the form (1). In [8] D. R. Wilken and R. Hornblower showed that each extreme point of $\mathcal{HC}$ is a support point of $\mathcal{C}$.

A natural question arises as to whether the functions (1) are support points of $\mathcal{S}$ or extreme points of $\mathcal{HS}$. Each function $f_{xy}$ in (1) maps the open unit disk to the complement of a half-line. Let $\Gamma_{xy}$, the omitted half-line of $f_{xy}$, be oriented so that $\Gamma_{xy}$ is traversed from $P_{xy}$, the finite tip of $\Gamma_{xy}$, to $\infty$. A computation shows that $|\arg(-x/y)|$ is the angle between the tangent vector to $\Gamma_{xy}$ and the radius vector to $\Gamma_{xy}$ at $P_{xy}$. It is easily seen that the angle between the tangent vector to $\Gamma_{xy}$ and the radius vector to $\Gamma_{xy}$ decreases (monotonically) to 0 as $\Gamma_{xy}$ is traversed (monotonically) from $P_{xy}$ to $\infty$. Thus, if $\pi/4 < |\arg(-x/y)| < \pi$, then $f_{xy}$ can be neither a support point of $\mathcal{S}$ nor an extreme point of $\mathcal{HS}$ (because $\Gamma_{xy}$ fails to satisfy the $\pi/4$-property). If $|\arg(-x/y)| = 0$, i.e., if $-x = y$, then $f_{xy}$ is the Koebe function $k_y$ and is both a support point of $\mathcal{S}$ and an extreme point of $\mathcal{HS}$. In the remaining case–$0 < |\arg(-x/y)| \leqslant \pi/4$–$\Gamma_{xy}$ does not violate the $\pi/4$-property. We will show for $0 < |\arg(-x/y)| \leqslant \pi/4$ that $f_{xy}$ is both a support point of $\mathcal{S}$ and an extreme point of $\mathcal{HS}$.

To prove the main result of this paper, we recall the bound on $|\arg f'(z_0)|$ for $f$ in $\mathcal{S}$ given by G. M. Goluzin [6, p. 115]. Namely, Goluzin showed that if $f \in \mathcal{S}$, then

$$|\arg f'(z_0)| \leqslant 4 \arcsin|z_0|, \qquad |z_0| \leqslant \frac{1}{\sqrt{2}}. \tag{2}$$

We now prove

THEOREM. *Let $f_{xy}$ be given by* (1). *If* $0 < |\arg(-x/y)| \leqslant \pi/4$, *then $f_{xy}$ is both a support point of* $\mathcal{S}$ *and an extreme point of* $\mathcal{HS}$.

PROOF. If we differentiate $f_{x,y}$ and then evaluate at $z_0$, we have

$$f'_{x,y}(z_0) = \frac{1 - xz_0}{(1 - yz_0)^3}.$$

An easy argument shows for $0 < |z_0| < 1$ that

$$|\arg f'_{x,y}(z_0)| \leq 4 \arcsin|z_0| \tag{3}$$

and that equality occurs in (3) if and only if

$$\arg xz_0 = -\arccos|z_0|, \quad \arg yz_0 = \arccos|z_0| \tag{4}$$

or

$$\arg xz_0 = \arccos|z_0|, \quad \arg yz_0 = -\arccos|z_0|. \tag{5}$$

If (4) holds, then $\arg f'_{x,y}(z_0) = 4 \arcsin|z_0|$ and if (5) holds, then $\arg f'_{x,y}(z_0) = -4 \arcsin|z_0|$. We note that for each pair $\{x, y\}$, $|x| = |y| = 1$, $x^2 \neq y^2$, there exists a unique $z_0$, $0 < |z_0| < 1$, such that exactly one of (4) or (5) holds.

Let $0 < |\arg(-x/y)| \leq \pi/4$ and suppose $z_0$ satisfies (4). Then (4) implies $0 < |z_0| \leq \sin \pi/8$. Goluzin's bound (2) on $|\arg f'(z_0)|$ implies that the region of variability of $f'(z_0)$ for $f$ in $\mathcal{S}$ lies in a closed sector in the closed right half-plane. Together (2)–(4) imply that $f'_{x,y}(z_0)$ lies on the upper edge of the region of variability of $f'(z_0)$ for $f$ in $\mathcal{S}$. By rotating the region of variability of $f'(z_0)$ for $f$ in $\mathcal{S}$ we can realize a continuous linear functional $J_{x,y}$ whose real part is maximized over $\mathcal{S}$ by $f_{x,y}$; namely

$$J_{x,y}g = -e^{i(\pi/2 - 4 \arcsin|z_0|)}g'(z_0).$$

Similarly, if $0 < |\arg(-x/y)| \leq \pi/4$ and $z_0$ satisfies (5), then $f_{x,y}$ maximizes $\operatorname{Re} J_{x,y}$ over $\mathcal{S}$ where

$$J_{x,y}g = -e^{-i(\pi/2 - 4 \arcsin|z_0|)}g'(z_0).$$

We will show now that if $0 < |\arg(-x/y)| < \pi/4$, then $\operatorname{Re} J_{x,y}$ is uniquely maximized over $\mathcal{S}$ by $f_{x,y}$, and if $|\arg(-x/y)| = \pi/4$, then $\operatorname{Re} J_{x,y}$ is maximized over $\mathcal{S}$ (only) by $f_{x,y}$ and $f_{y,x}$. The lemma will then imply if $0 < |\arg(-x/y)| \leq \pi/4$, then $f_{x,y}$ is an extreme point of $\mathcal{HS}$.

As in the first part, we can see that if $0 < |z_0| \leq \sin \pi/8$ and $f^*$ in $\mathcal{S}$ maximizes (minimizes) $\arg f'(z_0)$ over $\mathcal{S}$, then $f^*$ is a support point of $\mathcal{S}$ and, hence, in particular, a slit mapping. Goluzin's argument [**6**, p. 115], which shows that (2) is sharp, also shows that for $0 < |z_0| \leq 1/\sqrt{2}$ there exists a unique slit mapping which maximizes (minimizes) $\arg f'(z_0)$ over $\mathcal{S}$.

Let $0 < |\arg(-x/y)| < \pi/4$ and let $z_0$ satisfy (4). Since determining the functions which maximize $\operatorname{Re} J_{x,y}$ over $\mathcal{S}$ is equivalent to determining the functions which maximize $\arg f'(z_0)$ over $\mathcal{S}$, we conclude from the above that $\operatorname{Re} J_{x,y}$ is uniquely maximized over $\mathcal{S}$ by $f_{x,y}$. Similarly, if $0 < |\arg(-x/y)| < \pi/4$ and $z_0$ satisfies (5), then $\operatorname{Re} J_{x,y}$ is uniquely maximized over $\mathcal{S}$ by $f_{x,y}$.

Let $|\arg(-x/y)| = \pi/4$ and let $z_0$ satisfy (4) or (5). It is easily seen, from (2)–(5) that one of $f_{x,y}$ and $f_{y,x}$ maximizes $\arg f'(z_0)$ over $\mathcal{S}$ and the other minimizes $\arg f'(z_0)$ over $\mathcal{S}$. Since, in this case, we have $|z_0| = \sin \pi/8$, it follows that

$J_{x,y} = J_{y,x}$. Thus, determining the functions which maximize Re $J_{x,y}$ over $\mathcal{S}$ is equivalent to determining the functions which maximize or minimize $\arg f'(z_0)$ over $\mathcal{S}$. Consequently, Re $J_{x,y}$ is maximized over $\mathcal{S}$ (only) by $f_{x,y}$ and $f_{y,x}$.

REMARK. For the functions $f_{x,y}$ with $|\arg(-x/y)| = \pi/4$, the known bound of $\pi/4$ for the acute angle between the omitted arc of a support point of $\mathcal{S}$ and the radius vector is achieved (at the finite tip).

ACKNOWLEDGEMENT. This paper forms part of the author's doctoral thesis written at the State University of New York at Albany. The author would like to thank Professor Louis Brickman for his guidance and direction.

## BIBLIOGRAPHY

1. L. Brickman, *Extreme points of the set of univalent functions*, Bull. Amer. Math. Soc. **76** (1970), 372–374. MR **41** #448.

2. L. Brickman, T. H. MacGregor and D. R. Wilken, *Convex hulls of some classical families of univalent functions*, Trans. Amer. Math. Soc. **156** (1971), 91–107. MR **43** #494.

3. L. Brickman and D. R. Wilken, *Support points of the set of univalent functions*, Proc. Amer. Math. Soc. **42** (1974), 523–528.

4. J. E. Brown, *Geometric properties of a class of support points of univalent functions* (manuscript).

5. N. Dunford and J. T. Schwartz, *Linear operators*. Part I: *General theory*, Pure and Appl. Math., Vol. 7, Interscience, New York, 1958. MR **22** #8032.

6. G. M. Goluzin, *Geometric theory of functions of a complex variable*, GITTL, Moscow, 1952; English transl., Transl. Math. Monographs, Vol. 26, Amer. Math. Soc., Providence, R.I., 1969. MR **15** #112; **40** #308.

7. E. Grassman, W. Hengartner and G. Schober, *Support points of close-to-convex functions*, Canad. Math. Bull. **19** (1976), 177–179.

8. R. Hornblower and D. R. Wilken, *On the support points of close-to-convex functions* (manuscript).

9. W. E. Kirwan and R. W. Pell, *Extremal properties for a class of slit conformal mappings* (manuscript).

10. A. Pfluger, *Lineare Extremalprobleme bei schlichten Funktionen*, Ann. Acad. Sci. Fenn. Ser. AI **489** (1971), 32 pp. MR **45** #5337.

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT ALBANY, ALBANY, NEW YORK 12222

*Current address*: Department of Mathematics, Texas Tech University, Lubbock, Texas 79409