

HOLOMORPHIC MAPS THAT EXTEND TO AUTOMORPHISMS OF A BALL

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ABSTRACT. It is proved, under hypotheses that may be close to minimal, that certain types of biholomorphic maps of subregions of the unit ball in \mathbb{C}^n have the extension property to which the title alludes.

Let B (or B_n , when necessary) denote the open unit ball of \mathbb{C}^n . Thus $z = (z_1, \dots, z_n) \in B$ provided that $|z| < 1$, where $|z| = \langle z, z \rangle^{1/2}$ and $\langle z, w \rangle = \sum z_j \bar{w}_j$. An *automorphism* of B , i.e., a member of $\text{Aut}(B)$, is, by definition, a holomorphic map of B onto B that is one-to-one, and whose inverse is therefore also holomorphic. The sphere that bounds B is denoted by S .

The following extension theorem will be proved.

THEOREM. Assume that $n > 1$, and that

- (a) Ω_1 and Ω_2 are connected open subsets of B ,
- (b) for $j = 1, 2$, Γ_j is an open subset of S such that $\Gamma_j \subset \partial\Omega_j$,
- (c) F is a holomorphic one-to-one map of Ω_1 onto Ω_2 , and
- (d) there is a point $\alpha \in \Gamma_1$, not a limit point of $B \cap \partial\Omega_1$, and a sequence $\{a_i\}$ in Ω_1 , converging to α , such that $\{F(a_i)\}$ converges to a point $\beta \in \Gamma_2$, not a limit point of $B \cap \partial\Omega_2$.

Then there exists $\Phi \in \text{Aut}(B)$ such that $\Phi(z) = F(z)$ for all $z \in \Omega_1$.

The relation of this theorem to earlier results will be discussed after its proof.

The proof will use the following well-known facts.

(I) If $F: B_k \rightarrow B_n$ is holomorphic, and $F(0) = 0$, then $|F(z)| < |z|$ for all $z \in B_k$, and the linear operator $F'(0)$ (the Fréchet derivative of F at 0) maps B_k into B_n .

(II) If, in addition, $k = n$, then the Jacobian JF of F satisfies $|(JF)(0)| < 1$; equality holds only when F is a unitary operator on \mathbb{C}^n .

(III) If $F \in \text{Aut}(B)$ and $F(0) = 0$, then F is unitary.

Here is a brief indication of how these are proved. For unit vectors u and v in \mathbb{C}^k and \mathbb{C}^n , respectively, the classical Schwarz lemma applies to the function g defined by

$$g(\lambda) = \langle F(\lambda u), v \rangle, \quad (\lambda \in \mathbb{C}, |\lambda| < 1). \quad (1)$$

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Thus $|g(\lambda)| < |\lambda|$ for all eligible u, v , which leads to $|F(z)| < |z|$, and $|g'(0)| < 1$, which completes (I), since

$$g'(0) = \langle F'(0)u, v \rangle. \tag{2}$$

Since (I) implies that no eigenvalue of $F'(0)$ exceeds 1 in absolute value, it follows that

$$|(JF)(0)| = |\det F'(0)| < 1. \tag{3}$$

If $|(JF)(0)| = 1$, then the linear operator $F'(0)$ preserves volume, and maps B into B , hence is a unitary operator U . From this it follows easily (by considering iterates of $U^{-1}F$) that $F = U$.

To prove (III), apply (II) to F as well as to F^{-1} .

The following lemma contains the essence of the proof of the theorem. To state it, we introduce the notation (for $z \in \mathbb{C}^n$)

$$D_z = \{\lambda z: \lambda \in \mathbb{C}, \lambda z \in B\}. \tag{4}$$

Thus, when $z \neq 0$, D_z is the disc that is the intersection with B of the complex line through 0 and z .

LEMMA. Assume that

- (i) Ω_1 and Ω_2 are connected open sets in B ,
- (ii) $0 \in \Omega_1, 0 \in \Omega_2$.
- (iii) F is a holomorphic one-to-one map of Ω_1 onto Ω_2 , with $F(0) = 0$, and
- (iv) there is a nonempty open set $V \subset \Omega_1$, such that $D_z \subset \Omega_1$ and $D_{F(z)} \subset \Omega_2$ for every $z \in V$.

Then there is a unitary transformation U on \mathbb{C}^n such that $F(z) = Uz$ for all $z \in \Omega_1$.

PROOF OF THE LEMMA. If $z \in V$, then D_z lies in the domain of F . Identifying D_z with B_1 , we see from fact (I) (the case $k = 1$), that $|w| < |z|$, where $w = F(z)$. But D_w lies in the domain of F^{-1} , and the same argument shows that $|z| < |w|$. Thus $|F(z)|^2 = |z|^2$ for all $z \in V$. Both of these functions are real-analytic, hence they are equal in all of Ω_1 . In particular, choosing $r > 0$ so small that $rB \subset \Omega_1$, we see that $|F(z)| = |z|$ for all $z \in rB$. An appropriately scaled version of fact (III) shows now that F is unitary.

PROOF OF THE THEOREM. Let $\{a_i\}$ be as in assumption (d), put $b_i = F(a_i)$, and choose $u_i \in S, v_i \in S$, so that

$$a_i = |a_i|u_i, \quad b_i = |b_i|v_i, \quad (i = 1, 2, 3, \dots). \tag{5}$$

The geometric information contained in (d) shows that there exists $t < 1$ such that, setting

$$E_t(\xi) = \{z \in B: t < \operatorname{Re}\langle z, \xi \rangle\}, \quad (\xi \in S), \tag{6}$$

we have $a_i \in E_t(u_i) \subset \Omega_1$, and $b_i \in E_t(v_i) \subset \Omega_2$ for all sufficiently large i , say $i > i_0$.

If $a \in B \setminus \{0\}$, let P denote the orthogonal projection of \mathbb{C}^n onto the one-dimensional subspace spanned by a , put $Q = I - P$, and define

$$\varphi_a(z) = \frac{a - Pz - (1 - |a|^2)^{1/2}Qz}{1 - \langle z, a \rangle}, \quad (z \in \bar{B}). \tag{7}$$

Then (see [4], for instance) $\varphi_a \in \text{Aut}(B)$ and $\varphi_a^{-1} = \varphi_a$. Define

$$G_i = \varphi_{b_i} \circ F \circ \varphi_{a_i}, \quad (i > i_0). \tag{8}$$

Each G_i is a holomorphic one-to-one map of $\Omega_1^i = \varphi_{a_i}(\Omega_1)$ onto $\Omega_2^i = \varphi_{b_i}(\Omega_2)$, and $G_i(0) = 0$.

If $a = |a|\xi$, then $\langle Pz, \xi \rangle = \langle z, \xi \rangle \xi$, hence

$$\langle \varphi_a(z), \xi \rangle = (|a| - \langle z, \xi \rangle) / (1 - |a|\langle z, \xi \rangle). \tag{9}$$

If $t < |a|$, it follows that $\varphi_a(E_t(\xi))$ contains all $z \in B$ with

$$\text{Re}\langle z, \xi \rangle < (|a| - t) / (1 - |a|t). \tag{10}$$

Since $|a_i| \rightarrow 1$ and $|b_i| \rightarrow 1$, and since the right side of (10) tends to 1 as $|a|$ tends to 1, there is a sequence $\{r_i\}$, $r_i < 1$, such that $r_i \rightarrow 1$ as $i \rightarrow \infty$, and such that

$$z \in B, \text{Re}\langle z, u_i \rangle < r_i \text{ implies } z \in \Omega_1^i, \tag{11}$$

$$w \in B, \text{Re}\langle z, v_i \rangle < r_i \text{ implies } w \in \Omega_2^i. \tag{12}$$

By (11), $r_i B \subset \Omega_1^i$, the domain of G_i . Since $G_i(0) = 0$, fact (II) gives $|(JG_i)(0)| < r_i^{-n}$. In the same way, (12) leads to $|(JG_i^{-1})(0)| < r_i^{-n}$, so that $|(JG_i)(0)| > r_i^n$. A normal family argument shows now that a subsequence of $\{G_i\}$ converges, uniformly on compact subsets of B , to a holomorphic map of B into B that fixes 0 and whose Jacobian at 0 has absolute value 1. By fact (II), this limit map is unitary. Call it U .

Let V_i be the set of all $p \in B$ such that

$$D_z \subset \Omega_1^i \quad \text{and} \quad D_{Uz} \subset \Omega_2^i \tag{13}$$

for all z in some neighborhood of p .

Now fix ϵ , $0 < \epsilon < 1/10$. Using (11)–(13), we see that there is an index i , fixed from now on, such that

$$|G_i(z) - Uz| < \epsilon \quad \text{whenever } |z| < 1 - \epsilon, \tag{14}$$

and such that V_i contains a ball of radius 2ϵ , whose center p satisfies $|p| < 1 - 3\epsilon$. To see in more detail that this can indeed be done, note that when r_i is sufficiently close to 1, there exists a large set of points $\xi \in S$ such that $|\langle \xi, u_i \rangle| < r_i$ and $|\langle \xi, U^{-1}v_i \rangle| < r_i$. For any such ξ , $D_\xi \subset \Omega_1^i$ and $D_{U\xi} \subset \Omega_2^i$, thus $\lambda\xi \in V_i$ if $0 < |\lambda| < 1$.

Thus $D_z \subset \Omega_1^i$ if $|z - p| < 2\epsilon$, and $D_w \subset \Omega_2^i$ if $|w - Up| < 2\epsilon$. If $|z - p| < \epsilon$, and $w = G_i(z)$, it follows that $D_w \subset \Omega_2^i$ because

$$|w - Up| < |G_i(z) - Uz| + |z - p| < 2\epsilon. \tag{15}$$

The lemma applies therefore to G_i and shows that G_i is (the restriction of) a unitary operator. Since (8) gives

$$F = \varphi_{b_i} \circ G_i \circ \varphi_{a_i}, \tag{16}$$

the theorem is proved.

REMARKS. (i) Let Ω be a connected open subset of B such that $\bar{\Omega}$ contains an open subset Γ of S . If F is a nonconstant C^1 -map of $\bar{\Omega}$ into \bar{B} that is holomorphic in Ω and carries Γ into S , then $F \in \text{Aut}(B)$. This was proved by Pinčuk [6, p. 381],

who extended an earlier version due to Alexander [1] in which C^∞ was assumed in place of C^1 .

This Alexander-Pinčuk result is a fairly direct corollary of the present theorem. If $F \in C^1(\bar{\Omega})$ satisfies the Alexander-Pinčuk hypotheses, it is not hard to show (see Fornaess [3, p. 549] or Pinčuk [6, p. 378]) that JF vanishes at no point of Γ . The inverse function theorem implies then that the hypotheses of the present theorem hold.

(ii) In Alexander's proof [2] that every proper holomorphic map of B into B is in $\text{Aut}(B)$ when $n > 1$, his appeal to Fefferman's theorem can be replaced by the one proved in the present paper. Consequently, there exists now a much more elementary proof of the proper mapping theorem for B .

(iii) It is quite possible that the present theorem remains true if B is replaced by strictly pseudoconvex domains with real-analytic boundaries (as Pinčuk did in the C^1 -case [7]), but an entirely different proof would have to be found; Rosay [8] (strengthening a result of Wong [9]) proved that if some boundary point ξ of a bounded domain $\Omega \subset \mathbb{C}^n$ is a point of strict pseudoconvexity, and if there exist automorphisms T_k of Ω such that $\lim_{k \rightarrow \infty} T_k(p) = \xi$ for some $p \in \Omega$, then Ω is biholomorphically equivalent to B .

In other strictly pseudoconvex bounded domains there are thus insufficiently many automorphisms to imitate the proof that works in B .

(iv) If $\xi \in S$ and $\Omega = B \cap \{z: |\xi - z| < 1\}$; in other words, if $\Omega = B \cap (\xi + B)$, then the map $z \rightarrow \xi - z$ of Ω onto Ω demonstrates the relevance of the assumptions concerning the location of the points α and β in our theorem.

REFERENCES

1. H. Alexander, *Holomorphic mappings from the ball and polydisc*, Math. Ann. **209** (1974), 249–256.
2. ———, *Proper holomorphic mappings in \mathbb{C}^n* , Indiana Univ. Math. J. **26** (1977), 137–146.
3. J. E. Fornaess, *Embedding strictly pseudoconvex domains in convex domains*, Amer. J. Math. **98** (1976), 529–569.
4. A. Nagel and W. Rudin, *Moebius-invariant function spaces on balls and spheres*, Duke Math. J. **43** (1976), 841–865.
5. S. I. Pinčuk, *On proper holomorphic mappings of strictly pseudoconvex domains*, Siberian Math. J. **15** (1974), 644–649.
6. ———, *On the analytic continuation of holomorphic mappings*, Math. USSR-Sb. **27** (1975), 375–392.
7. ———, *Analytic continuation of mappings along strictly pseudoconvex hypersurfaces*, Soviet Math. Dokl. **18** (1977), 1237–1240.
8. J. P. Rosay, *Sur une caractérisation de la boule parmi les domaines de \mathbb{C}^n par son groupe d'automorphismes*, Ann. Inst. Fourier **29** (1979), 91–97.
9. B. Wong, *Characterization of the unit ball in \mathbb{C}^n by its automorphism group*, Invent. Math. **41** (1977), 253–257.

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