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ON THE HOMOTOPY AND COHOMOLOGY OF THE CLASSIFYING SPACE OF RIEMANNIAN FOLIATIONS[1]

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ABSTRACT. Let G be a closed subgroup of the general linear group. Let $B\tilde{\Gamma}_G^q$ be the classifying space for G-foliated microbundles of rank q. (The G-foliation is not assumed to be integrable.) The homotopy fiber $F\tilde{\Gamma}_G^q$ of the classifying map ν: $B\tilde{\Gamma}_G^q \to BG$ is shown to be $(q-1)$-connected. For the orthogonal group, this implies $FR\Gamma^q$ is $(q-1)$-connected. The indecomposable classes in $H^*(RW_q)$ therefore are mapped to linearly independent classes in $H^*(FR\Gamma^q)$; the indecomposable variable classes are mapped to independently variable classes. Related results on the homotopy groups $\pi_*(FR\Gamma^q)$ also follow.

1. The main theorem. Let $BR\Gamma^q$ be the Haefliger classifying space for Riemannian foliations, $BO(q)$ the classifying space for $O(q)$-bundles and ν: $BR\Gamma^q \to BO(q)$ the map classifying the normal bundle of the universal $R\Gamma^q$-structure on $BR\Gamma^q$ [3]. Let $FR\Gamma^q$ be the homotopy theoretic fiber of ν. $H^*(\ )$ will denote singular cohomology with real coefficients. In this note we show

THEOREM 1.1. *$FR\Gamma^q$ is $(q-1)$-connected.*

This implies there is a section of ν over the q-skeleton of $BO(q)$, so ν^*: $H^q(BO(q)) \to H^q(BR\Gamma^q)$ is injective. On the other hand, the vanishing Theorem of J. Pasternack [9] implies ν^*: $H^{q+1}(BO(q)) \to H^{q+1}(BR\Gamma^q)$ is the zero map. Theorem 1.1 is therefore the best result possible for $q = 4k + 3$. For other q, it would be interesting to know whether $FR\Gamma^q$ has higher connectivity.

Theorem 1.1 is a special case of a more general result. Let $G \subseteq Gl(q, \mathbf{R})$ be a closed subgroup. A foliation on a manifold M is said to be a G-foliation [1], [7] if there is given

(i) a model manifold B of dimension q with a G-structure on TB,

(ii) an open covering $\{U_\alpha\}$ of M and local submersions ϕ_α: $U_\alpha \to B$ defining the foliation such that the transition functions $\gamma_{\alpha\beta}$ are local G-morphisms of B.

A G-foliation is *integrable* if it is modeled on $\mathbf{R}^q$ with the flat G-structure.

A classifying space for G-foliations is constructed as follows: Let $\mathfrak{N}(G, \mathbf{R}^q)$ denote the total space of the sheaf of local C^∞-sections of the bundle $\mathbf{R}^q \times Gl(q, \mathbf{R})/G \to \mathbf{R}^q$. This is a (non-Hausdorff) C^∞-manifold, with a canonical G-structure. Let $\mathcal{G}_G$ be the pseudogroup of all local, C^∞, G-diffeomorphisms of

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$\mathfrak{N}(G, \mathbf{R}^q)$ and let $\tilde{\Gamma}_G^q$ be its associated topological groupoid [1, §2], [3]. Let $B\tilde{\Gamma}_G^q$ be the Haefliger classifying space of $\tilde{\Gamma}_G^q$-structures. For $G = O(q)$, we have $BR\Gamma^q = B\tilde{\Gamma}_{O(q)}^q$ and, in general, $B\tilde{\Gamma}_G^q$ is the classifying space of G-foliations.

There is a natural map ν: $B\tilde{\Gamma}_G^q \to BG$ classifying the normal bundle of the $\tilde{\Gamma}_G^q$-structure on $B\tilde{\Gamma}_G^q$. Let $F\tilde{\Gamma}_G^q$ be the homotopy theoretic fiber of ν.

THEOREM 1.1$'$. *$F\tilde{\Gamma}_G^q$ is $(q-1)$-connected.*

For $G = O(q)$ we recover Theorem 1.1.

There is also a classifying space for integrable G-foliations, denoted by $B\Gamma_G^q$. We let $F\Gamma_G^q$ denote the homotopy theoretic fiber of ν: $B\Gamma_G^q \to BG$. When $G = Sl(q, \mathbf{R})$, one can show $B\tilde{\Gamma}_{Sl}^q \simeq B\Gamma_{Sl}^q$ [1, Remark 4.2], recovering from Theorem 1.1$'$ Haefliger's result that $F\Gamma_{Sl}^q$ is $(q-1)$-connected.

2. Applications. In this section, we give some consequences of Theorem 1.1$'$. The proofs of the propositions stated use Sullivan's theory of minimal models [10], and are given in [5].

Theorem 1.1 implies there are many nontrivial Whitehead products in $\pi_*(BR\Gamma^q)$ and that many of the secondary characteristic classes map injectively into $H^*(FR\Gamma^q)$. To be precise, let $q' = [q/2]$ and $W(\mathfrak{so}(q))_{q'}$ denote the truncated Weil algebra for the orthogonal Lie algebra [7]. The Chern-Weil construction gives a characteristic map Δ_*: $H^*(W(\mathfrak{so}(q))_{q'}) \to H^*(FR\Gamma^q)$. Let $k = [q/4] + 1$ and $m = [(q-1)/2]$. The set of invariants factors as

$$H^*(W(\mathfrak{so}(q))_{q'}) \simeq A \otimes \Lambda(y_k, \ldots, y_m),$$

where A is an algebra with all products zero and the second factor is the exterior algebra on generators y_j of degree $4j - 1$. The algebra A has an explicit basis, given by 1 and the cocycles $y_I p_J \in W(\mathfrak{so}(q))_{q'}$ where

$$y_I p_J = y_{i_1} \cdots y_{i_s} p_1^{j_1} \cdots p_{k-1}^{j_{k-1}},$$

$1 \leq i_1 < \cdots < i_s < k$ and $l < i_1 \Rightarrow j_l = 0$, and $\deg p_{i_1} p_J > q$, $\deg p_J \leq q$.

For q even, additional cocycles involving the Euler class must be added to this list [8]. A basis element $y_I p_J \in A$ is said to be variable if $\deg y_{i_1} p_J = 2q' + 1$.

Let $V \subseteq H^*(W(\mathfrak{so}(q))_{q'})$ be the subspace given by the direct sum

$$V = A \otimes 1 \oplus 1 \otimes \Lambda(y_k, \ldots, y_m).$$

PROPOSITION 2.1. *Δ_*: $V \to H^*(FR\Gamma^q)$ is injective, and the variable basis elements in V are mapped to independently variable classes in $H^*(FR\Gamma^q)$.*

The first statement follows from Theorem 1.1 and the results of F. Kamber and Ph. Tondeur [6, Theorem 6.52]. The variability follows from the examples of C. Lazarov and J. Pasternack [8, Theorem 3.6] combined with Theorem 1.1. Details can be found in [5].

Similar results concerning the homotopy of $FR\Gamma^q$ can be shown. Set $\pi^*(FR\Gamma^q) = \mathrm{Hom}(\pi_*(FR\Gamma^q), \mathbf{R})$. Let $\langle y_k, \ldots, y_m \rangle$ denote the real vector space spanned by $\{y_k, \ldots, y_m\}$. In [5], a vector-space map $h^\# \circ \zeta$: $H^*(W(\mathfrak{so}(q))_{q'}) \to \pi^*(FR\Gamma^q)$ is defined, for which

PROPOSITION 2.2. *$h^\# \circ \zeta$: $A \oplus \langle y_k, \dots, y_m \rangle \to \pi^*(FR\Gamma^q)$ is injective and the variable basis elements of A are mapped to independently variable classes.*

For any commutative cochain algebra $\mathcal{Q}$ there is a vector space $\pi^*(\mathcal{Q})$, the dual homotopy of $\mathcal{Q}$, constructed by choosing a minimal model $\mathfrak{M} \to \mathcal{Q}$, and setting $\pi^*(\mathcal{Q}) = \mathfrak{M}^*/(\mathfrak{M}^+ \cdot \mathfrak{M}^+)$ **[10]**. The algebra A has trivial products and differential, so for $q = 4, 6$ or $\geqslant 8$ the vector space $\pi^*(A)$ is of finite type but not finite dimensional. There is induced a map $\Delta^\#$: $\pi^*(A) \to \pi^*(FR\Gamma^q)$, extending $h^\# \circ \zeta$, for which we have **[5]**

PROPOSITION 2.3. *$\Delta^\#$: $\pi^*(A) \oplus \langle y_k, \dots, y_m \rangle \to \pi^*(FR\Gamma^q)$ is injective and the variable classes are mapped to independently variable classes.*

The following proposition gives our final remark on the homotopy of $F\tilde{\Gamma}^q_G$. The proof is obvious, using minimal models.

PROPOSITION 2.4. *Let X be an n-connected space, $n \geqslant 1$. Then the rational Hurewicz map $\mathcal{H}$: $\pi_m(X) \otimes \mathbf{Q} \to H_m(X; \mathbf{Q})$ is an isomorphism for $m \leqslant 2n$ and an epimorphism for $m = 2n + 1$.*

COROLLARY 2.5. *$\mathcal{H}$: $\pi_m(F\tilde{\Gamma}^q_G) \otimes \mathbf{Q} \to H_m(F\tilde{\Gamma}^q_G; \mathbf{Q})$ is an isomorphism for $m \leqslant 2q - 2$ and an epimorphism for $m = 2q - 1$.*

3. Proof of Theorem 1.1′. Let $X \subseteq \mathbf{R}^q$ be an open subset homotopic to S^n. When n is zero, we consider S^0 to consist of a single point. Then $\pi_n(F\tilde{\Gamma}^q_G) \cong [X, F\tilde{\Gamma}^q_G]$, the set of homotopy classes of maps f: $X \to F\tilde{\Gamma}^q_G$. By the Gromov-Phillips-Haefliger Theorem [2], there is a bijection between $[X, F\tilde{\Gamma}^q_G]$ and the set of integrable homotopy classes of G-foliations on X with trivial G-structure. We will show two such foliations on X are integrably homotopic.

Recall that two codimension q G-foliations $\mathcal{F}_0$, $\mathcal{F}_1$ on X are integrably homotopic if there is a codimension q G-foliation $\mathcal{F}$ on $X \times [0, 1]$ such that the slices i_t: $X \times \{t\} \to X \times [0, 1]$ are transverse to $\mathcal{F}$ for all t, and induce $\mathcal{F}_t$ for $t = 0, 1$.

Fix an integer n with $0 \leqslant n < q$. Let $(\theta, r) \in \mathbf{R}^{n+1}$ be polar coordinates, with $\theta \in S^n$ and $r \in \mathbf{R}$. For any $a, b \in \mathbf{R}$ with $0 \leqslant a < b$, define

$$B(a, b) = \{(\theta, r) \in \mathbf{R}^{n+1} | a < r < b\} \times \mathbf{R}^{q-n-1}.$$

Set $X = B(0, 1)$; then $X \subseteq \mathbf{R}^q$ is open and homotopic to S^n.

A codimension q G-foliation on X must be the point foliation with a G-structure on the tangent bundle TX. The tangent bundle is trivial, so the G-structure is characterized by a smooth map α: $X \to Y$, where Y is the coset space $Gl(q, \mathbf{R})/G$. We denote by (X, α) the G-foliation on X with characteristic map α. The G-structure on (X, α) is trivial if α is homotopic to the constant map with image the identity coset of Y. For two G-foliations (X, α_0) and (X, α_1) with trivial G-structures, it is apparent that α_0 and α_1 are homotopic.

To prove the theorem, it will suffice to show that if α_0 and α_1 are homotopic, then there is an integrable homotopy from (X, α_0) to (X, α_1). To do this, we will construct three integrable homotopies, on $X \times [0, 1]$, $X \times [1, 2]$ and $X \times [2, 3]$ which combine to give the desired integrable homotopy.

Step 1. Choose a monotone, C^∞-function

$$\phi\colon [0,1] \to [1/2, 1] \quad \text{with } \phi(t) = \begin{cases} 1 & \text{for } t < 1/4, \\ 1/2 & \text{for } t > 3/4. \end{cases}$$

Define $H\colon X \times [0, 1] \to X$ by

$$H_t(\theta, r, v) = (\theta, \phi(t)\cdot(r - 1/2) + 1/2, v).$$

For each t, H_t is a submersion; H_0 is the identity and H_1 maps X to a subannulus of X. Also, H_t is constant with respect to t for t near 0 or 1.

Define a G-structure on X by $\alpha_0' = \alpha_0 \circ H_1\colon X \times \{1\} \to Y$. Then the submersion $H\colon X \times I \to (X, \alpha_0)$ defines a G-foliation on $X \times [0, 1]$ which is an integrable homotopy from (X, α_0) to (X, α_0').

Step 2. Define $H''\colon X \times [2, 3] \to X$ by $H_t'' = H_{3-t}$. Define a G-structure on X by setting $\alpha_1' = \alpha_1 \circ H_2''$. Then the submersion $H''\colon X \times [2, 3] \to (X, \alpha_1)$ defines a G-foliation which is an integrable homotopy from (X, α_1') to (X, α_1).

Step 3. We next produce an integrable homotopy from (X, α_0') to (X, α_1') by constructing a G-foliation (X, α) and a submersion $H'\colon X \times [1, 2] \to (X, \alpha)$ so that $\alpha_0' = \alpha \circ H_1'$ and $\alpha_1' = \alpha \circ H_2'$.

Define functions f_0 and f_1 as follows

$$f_0\colon B(5/8, 1) \to B(0, 3/4) \quad \text{by } f_0(\theta, r, v) = (\theta, 2r - 5/4, v),$$
$$f_1\colon B(0, 3/8) \to B(1/4, 1) \quad \text{by } f_1(\theta, r, v) = (\theta, 2r + 1/4, v).$$

Note that f_0 maps $B(3/4, 1)$ to the image of H_1 and f_1 maps $B(0, 1/4)$ to the image of H_2''.

There are inclusions

$$i_0\colon S^n \times \{3/4\} \times \mathbf{R}^{n-q-1} \subseteq B(5/8, 1),$$
$$i_1\colon S^n \times \{1/4\} \times \mathbf{R}^{n-q-1} \subseteq B(0, 3/8)$$

and the composites $\alpha_0 \circ f_0 \circ i_0$ and $\alpha_1 \circ f_1 \circ i_1$ are homotopic by assumption. Therefore, there exists a smooth extension

$$\tilde{\alpha}\colon S^n \times [1/4, 3/4] \times \mathbf{R}^{n-q-1} = \overline{B(1/4, 3/4)} \to Y$$

of $\alpha_0 \circ f_0 \circ i_0 \cup \alpha_1 \circ f_1 \circ i_1$. We define a smooth map $\alpha\colon X \to Y$ by

$$\alpha = \begin{cases} \alpha_0 \circ f_0 & \text{on } B(3/4, 1), \\ \tilde{\alpha} & \text{on } \overline{B(1/4, 3/4)}, \\ \alpha_1 \circ f_1 & \text{on } B(0, 1/4). \end{cases}$$

Finally, we construct the submersion $H'\colon X \times [1, 2] \to X$. Choose a monotone, C^∞-function $\varphi\colon [1, 2] \to [0, 3]$ with

$$\varphi(t) = \begin{cases} 3 & \text{for } t < 5/4, \\ 0 & \text{for } t > 7/4. \end{cases}$$

Then H' at time t is given by

$$H_t'(\theta, r, v) = (\theta, 1/4(r + \varphi(t)), v).$$

The map H' has the effect of sliding the image of X from image H_1 to image H_2'' as t varies from 1 to 2.

Let $X \times [1, 2]$ have the G-structure defined by the submersion $H'\colon X \times [1, 2] \to (X, \alpha)$. This gives an integrable homotopy from $(X, \alpha \circ H_1')$ to $(X, \alpha \circ H_2')$. A straightforward check shows that $f_0 \circ H_1' = H_1$ and $f_1 \circ H_2' = H_2''$. This implies $\alpha_0' = \alpha \circ H_1'$ and $\alpha_1' = \alpha \circ H_2'$, which finishes Step 3 and the proof of Theorem 1.1′. □

REFERENCES

1. T. Duchamp, *Characteristic invariants of G-foliations*, Ph. D. thesis, University of Illinois, Urbana, Ill., 1976.

2. A. Haefliger, *Feuilletages sur les variétés ourvertes*, Topology **9** (1970), 183–194.

3. ______, *Homotopy and integrability*, Lecture Notes in Math., vol. 197, Springer-Verlag, Berlin and New York, 1971, pp. 133–163.

4. ______, *Whitehead products and differential forms*, Lecture Notes in Math., vol. 652, Springer-Verlag, Berlin and New York, pp. 13–24.

5. S. Hurder, *Dual homotopy invariants of G-foliations*, Ph. D. Thesis, University of Illinois, Urbana, Ill., 1980.

6. F. Kamber and Ph. Tondeur, *Non-trivial invariants of homogeneous foliated bundles*, Ann. Sci. École Norm. Sup. **8** (1975), 433–486.

7. ______, *G-foliations and their characteristic classes*, Bull. Amer. Math. Soc. **84** (1978), 1086–1124.

8. C. Lazarov and J. Pasternack, *Secondary characteristic classes for Riemannian foliations*, J. Differential Geometry **11** (1976), 365–385; *Residues and characteristic classes for Riemannian foliations*, J. Differential Geometry **11** (1976), 599–612.

9. J. Pasternack, *Foliations and compact Lie groups actions*, Comment. Math. Helv. **46** (1971), 467–477.

10. D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331.

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