KNOTS WITH HEEGAARD GENUS 2 COMPLEMENTS ARE INVERTIBLE

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Abstract. Let $K$ be a polyhedral oriented knot in $S^3$ and $N(K)$ be a regular neighborhood of $K$. If $S^3 \sim \tilde{N}(K)$ can be constructed by attaching a single 2-handle to a genus two handlebody, then there is a homeomorphism of $S^3$ onto itself mapping $K$ onto itself and reversing the orientation of $K$.

We prove the title. A somewhat more careful statement is the following.

Theorem. Let $K \subset S^3$ be a polyhedral knot and let $N(K)$ be a regular neighborhood of $K$. If $S^3 \sim \tilde{N}(K)$ can be constructed by attaching a single 2-handle to a genus two handlebody, then $K$ is invertible.

Proof. By a meridian of the knot $K$ we mean a polyhedral disk $D$ in $N(K)$ with $\partial D \subset \partial N(K)$ and $N(K) \sim N(D)$ is homeomorphic with a ball. ($N(D)$ is a regular neighborhood of $D$ in $N(K)$.) We show how to construct a self-homeomorphism of $S^3 \sim \tilde{N}(K)$ which maps the boundary of a meridian of the knot onto its inverse. It is easy to see that such a homeomorphism can be extended to an involution of $S^3$ taking the (oriented) knot to its inverse. Let $H_2$ be a genus two solid handlebody, let $D_1$ and $D_2$ be meridian disks for $H_2$ and let $\gamma$ be the simple closed curve on $H_2$ to which a 2-handle $B$ is attached to get $S^3 \sim \tilde{N}(K)$. Let $m$ be the boundary of a meridian of $K$. Without loss of generality we may assume that $m \subset \partial H_2 \sim \gamma$. Let $h: H_2 \rightarrow H_2$ be a rotation of $H_2$ through $180^\circ$ about its axis. (Think of the standard picture of $H_2$. The axis passes through both holes of the handlebody.) Now $h$ induces the symmetry $\eta$ as defined in [O & S, p. 248]. Thus $h$ can be assumed to map $\gamma$ and $m$ onto themselves while reversing their orientations (see also [B & H, §5]). The underlying reason for this is that the rotation inverts the Lickorish twists that generate the homeotopy group of the surface. Clearly then $h$ may be extended to a homeomorphism $\tilde{h}$ which maps $B$ onto itself while reversing orientation. This completes our proof.

Note. This result shows that knots such as $8_{20}$ and $10_{132}$ from the table of knots in [Rolf] are invertible. This does not appear evident from the presentations given. These knots are not torus knots or 2-bridge knots. The knot $8_{10}$ does not have a complement with Heegaard genus 2 but it is invertible. That it does not have such a complement follows from the fact that its second elementary ideal is proper [Fox]. Of course, the result above means that the noninvertible pretzel knots of Trotter

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[Trot] all have complements of Heegaard genus 3. It is certainly not easy to decide whether a given knot has a complement of Heegaard genus 2.

REFERENCES


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