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ON A PROBLEM OF LOHWATER ABOUT THE ASYMPTOTIC BEHAVIOUR OF NEVANLINNA'S CLASS

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ABSTRACT. Let $f(z)$ be meromorphic in $|z| < 1$ and let the radial limits $\lim_{r\to 1} f(re^{i\theta})$ exist and have modulus 1 for almost all $e^{i\theta} \in A = \{e^{i\theta}: \theta_1 < \theta < \theta_2\}$. If P is a singular point of $f(z)$ on A, then every value of modulus 1 which is not in the range of $f(z)$ at P is an asymptotic value of $f(z)$ at some point of each subarc of A containing the point P. This answers in the affirmative sense a question of A. J. Lohwater.

1. Introduction. In 1953 [**2**, Theorem 3], Lohwater proved the following result: Let $f(z)$ be meromorphic in the unit disk $D = \{z:|z| < 1\}$ with bounded characteristic in the sense of Nevanlinna, and let the radial limits $\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta})$ have modulus 1 for almost all $e^{i\theta} \in A = \{e^{i\theta}: \theta_1 < \theta < \theta_2\}$. If P is a singular point of $f(z)$ on A, then every value of modulus 1 which is not in the range of $f(z)$ at P is an asymptotic value of $f(z)$ at some point of each subarc of A containing the point P. He then asked as to whether this result is still true if $f(z)$ is not of bounded characteristic (see [**2**, p. 156]). The following theorem answers this question in the affirmative sense.

THEOREM 1. *Let $f(z)$ be meromorphic in D and let the radial limits* $\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta})$ *exist and have the modulus* 1 *for almost all $e^{i\theta} \in A = \{e^{i\theta}: \theta_1 < \theta < \theta_2\}$. If P is a singular point of $f(z)$ on A, then every value of modulus* 1 *which is not in the range of $f(z)$ at P is an asymptotic value of $f(z)$ at some point of each subarc of A containing the point P.*

2. Maximum principle. To prove Theorem 1, we shall need the following strong form of maximum principle due to Collingwood and Lohwater [**1**, Theorem 5.3].

LEMMA 1. *If $f(z)$ is analytic and bounded, $|f(z)| < M$, in D, and if the radial limit values $f(e^{i\theta})$ are in modulus not greater than $m < M$ almost everwhere on $|z| = 1$, then $|f(z)| < m$ everywhere in D, unless $f(z)$ is a constant of modulus m.*

3. Proof of Theorem 1. Let Q be a value of modulus 1 which is not in the range of $f(z)$ at P and let $g(z)$ be the function defined by $g(z) = 1/(f(z) - Q)$, for $z \in D$. Then there is a number $\delta > 0$ such that the function $g(z)$ is analytic in the set $D_\delta(P) = D \cap \{z: |z - P| < \delta\}$.

There are two cases to be considered: either the function $g(z)$ is bounded in the domain $D_\delta(P)$ or it is not. In the first case, we let $z = z(w)$ be a conformal

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mapping from the unit disk $D_w = \{w\colon |w| < 1\}$ onto the domain $D_\delta(P)$. Then by the Nevanlinna theory, see [1, p. 39], the functions $g(z(w))$ and $f(z(w))$ are both of bounded characteristic in D_w. In this case, the result follows from Lohwater's theorem [2, Theorem 3]. We may, therefore, assume that the function $g(z)$ is unbounded in $D_\delta(P)$.

For convenience, we write $P = e^{i\alpha}$, where $\theta_1 < \alpha < \theta_2$. From the hypothesis, we can choose two numbers α_1 and α_2 with $\theta_1 < \alpha_1 < \alpha < \alpha_2 < \theta_2$ and $\alpha_2 - \alpha_1 < \delta$ such that the radial limits $\lim_{r\to 1} f(re^{i\alpha_j}) = f(e^{i\alpha_j})$ exist for $j = 1, 2$. This in turn implies the following radial limits exist

$$\lim_{r\to 1} g(re^{i\alpha_j}) = 1/\left(f(e^{i\alpha_j}) - Q\right) = g(e^{i\alpha_j}), \quad \text{for } j = 1, 2. \tag{1}$$

Let R_j be the radius ending at $e^{i\alpha_j}$ and let S_j be the point of intersection of R_j with the boundary of $D_\delta(P)$ which lies within the disk D, where $j = 1, 2$. Denote by T the portion of the boundary of $D_\delta(P)$ between S_1 and S_2. If $f(z)$ has the radial limit Q on either R_1 or R_2, then there is nothing more to prove, so we may suppose that the radial limits of $f(z)$ along R_1 and R_2 are both different from Q. Then by (1) we find that the function $g(z)$ is bounded on the union $R_1 \cup T \cup R_2$, so that for some $M > 0$,

$$|g(z)| < M, \quad \text{for } z \in R_1 \cup T \cup R_2. \tag{2}$$

Let G be the domain bounded by $R_1 \cup T \cup R_2$ and the arc $A(\alpha_1, \alpha_2) = \{e^{i\theta}\colon \alpha_1 < \theta < \alpha_2\}$. If the function $g(z)$ is bounded in G, then by the same argument as before we know that the assertion follows from Lohwater's theorem. We may, therefore, assume that this function $g(z)$ is unbounded in G. Let $M_1 > M$ and let $H = \{z\colon |z| < 1 \text{ and } |g(z)| > M_1\}$ and $H_1 = H \cap G$. Since $g(z)$ is unbounded in G, the domain H_1 is not empty. Moreover, from (2) we can see that the boundary of H_1 is disjoint from the set $R_1 \cup T \cup R_2$. Let H_1^* be a component of H_1. We shall prove that $g(z)$ is unbounded in H_1^*. Suppose on the contrary that $|g(z)| < N$, for $z \in H_1^*$, where $N > M_1$. As before, let $z = z(w)$ be a conformal mapping from D_w onto H_1^*. For almost all $\theta \in [0, 2\pi]$, either $\lim_{r\to 1} |g(z(re^{i\theta}))| = M_1$ or $\lim_{r\to 1} g(z(re^{i\theta}))$ lies on the line L which is the image of the circle $|z| = 1$ under the mapping $\phi(z) = 1/(z - Q)$. The line L meets the circle $|z| = M_1$ from elementary considerations. Since $g(z(w))$ is a nonconstant bounded function on D_w, if we set $W = g(z(D_w))$ then there exist points ζ_1 and ζ_2 such that $\zeta_1 \in W$, ζ_2 is not in the closure of W, and the distance from ζ_1 to ζ_2 is less than half the distance from ζ_2 to $L \cup \{z\colon |z| = M_1\}$. Let $h(w) = 1/(g(z(w)) - \zeta_2)$. Then $h(w)$ is a bounded analytic function on D_w which assumes the value $1/(\zeta_1 - \zeta_2)$, but $h(w)$ has radial limits of modulus at most $1/(2|\zeta_1 - \zeta_2|)$ almost everywhere on $|w| = 1$, in violation of Lemma 1. It follows that $g(z)$ is unbounded on H_1^*. Based on this property, we shall construct a path Γ such that $\Gamma \subset G$ and $g(z)$ tends to infinity along this path Γ.

We begin by choosing $z_1 \in H_1^*$. Let r_2 and M_2 be two positive numbers such that $|z_1| < r_2 < 1$ and

$$M_1 > \max|g(z)|, \quad \text{for } z \in H_1^* \cap \{z\colon |z| < r_2\}.$$

Choose H_2^* a component of the domain $H_2 = \{z: z \in H_1^* \text{ and } |g(z)| > M_2\}$. Then H_2^* is not empty because $g(z)$ is unbounded in H_1^*. Choose a point $z_2 \in H_2^*$ and join z_1 to z_2 by an arc Γ_1 lying in H_1^*. Then clearly we have $|g(z)| > M_1$, for $z \in \Gamma_1$, where $\Gamma_1 \subset H_1^*$. Continuing this process, we can construct four sequences $\{H_n^*\}$, $\{z_n\}$, $\{\Gamma_n\}$, and $\{M_n\}$ such that

$H_1^* \supset H_2^* \supset \cdots, \Gamma_n \subset H_n^*$,

Γ_n joins z_n to z_{n+1}, $|z_n|\uparrow 1$, as $n \to \infty$, and

$|g(z)| > M_n$, for $z \in \Gamma_n$, where $M_n \uparrow \infty$, as $n \to \infty$.

Letting $\Gamma = \bigcup \Gamma_n$, we have that $\Gamma \subset G$ and that $g(z) \to \infty$ as $|z| \to 1$ along Γ. It then follows that the function $f(z)$ tends to the value Q along the path Γ.

By our construction, $\Gamma_A = \overline{\Gamma} \cap \{z: |z| = 1\} \subset \overline{A(\alpha_1, \alpha_2)}$ (where we use $\overline{E}$ to denote the closure of the set E). If Γ_A is a single point, then f has the asymptotic value Q at this point. But if Γ_A contains two points, then Γ_A is a closed subarc of $\overline{A(\alpha_1, \alpha_2)}$, and so the radial limit of f at almost every point of Γ_A must be Q, since each radius to a point of the interior of the arc Γ_A must meet the arc Γ at points arbitrarily close to $|z| = 1$. This completes the proof.

Finally, we remark that the hypotheses of Theorem 1 are insufficient to ensure that the function f is of bounded characteristic. Lohwater [2, p. 155] has constructed a function of unbounded characteristic having radial limits of modulus 1 at almost all points of the unit circle.

References

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