GROWTH OF POLYNOMIALS WITH ZEROS OUTSIDE A CIRCLE

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Abstract. Let \( P(z) \) be a polynomial of degree \( n \) having all its zeros in \( |z| > k > 1 \). For \( k = 1 \), it is known that
\[
\max_{|z| = R > 1} |P(z)| < \frac{R^n + 1}{2} \max_{|z| = 1} |P(z)|.
\]
In this paper we consider the case \( k > 1 \) and obtain a sharp result

Let \( P(z) \) be a polynomial of degree \( n \) such that \( \max_{|z| = 1} |P(z)| = 1 \), then
\[
\max_{|z| = R > 1} |P(z)| < R^n. \tag{1}
\]
Inequality (1) is a simple deduction from the Maximum Modulus Principle (see [4, p. 346] or [3, Vol. I, p. 137, problem III 269]). It was conjectured by Erdős and first proved by Lax [2] that, if \( P(z) \neq 0 \) in \( |z| < 1 \) then
\[
\max_{|z| = 1} |P'(z)| < n/2. \tag{2}
\]
Ankeny and Rivlin [1] used (2) to prove the following theorem.

**Theorem A.** If \( P(z) \) is a polynomial of degree \( n \) with \( \max_{|z| = 1} |P(z)| = 1 \) and \( P(z) \) has no zeros in the disk \( |z| < 1 \), then
\[
\max_{|z| = R > 1} |P(z)| < \frac{R^n + 1}{2}. \tag{3}
\]
The result is best possible and equality in (3) holds for \( P(z) = (z^n + 1)/2 \).

We shall generalize Theorem A by proving the following theorems.

**Theorem 1.** If \( P(z) \) is a polynomial of degree \( n \) with \( \max_{|z| = 1} |P(z)| = 1 \) and \( P(z) \) has no zeros in the disk \( |z| < k \) where \( K > 1 \), then
\[
\max_{|z| = R > 1} |P(z)| < \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + Rk)^n}. \tag{4}
\]
Theorem 1 is a generalization of Theorem A in a compact form but unfortunately with the exception of \( n = 1 \), (4) does not appear to be sharp for \( k > 1 \).

However, a precise estimate is given by the following theorem.

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Theorem 2. If \( P(z) \) is a polynomial of degree \( n \) with \( \max_{|z|=1} |P(z)| = 1 \) and \( P(z) \) has no zeros in the disk \( |z| < k \) where \( k > 1 \), then for \( R > k \) we have
\[
\max_{|z|=R} |P(z)| < \frac{R^n + k^n}{1+k^n} \quad \text{for } R > k^2
\]
and
\[
\max_{|z|=R} |P(z)| < \frac{(R+k)^n}{(1+k)^n} \quad \text{for } 1 < R < k^2.
\]

The result is best possible with equality in (5) for \( P(z) = \frac{z^n + k^n}{1+k^n} \) and in (6) for \( P(z) = \frac{z + k^n}{(1+k)^n} \).

For the proofs of these theorems we need the following Lemma.

Lemma. If \( P(z) \) is a polynomial of degree \( n \) with \( \max_{|z|=1} |P(z)| = 1 \), then
\[
|P(Re^{i\theta})| + |Q(Re^{i\theta})| < R^n + 1, \quad 0 < \theta < 2\pi,
\]
where \( Q(z) = z^n P(1/z) \) and \( R > 1 \).

Proof of the Lemma. Since \( |P(z)| < 1 \) for \( |z| = 1 \), therefore, if \( \alpha \) is a complex number such that \( |\alpha| > 1 \), it follows from Rouche’s theorem that the polynomial \( F(z) = P(z) - \alpha \) does not vanish inside the unit circle. Thus the polynomial
\[
G(z) = z^n F(1/z) = z^n P(1/z) - \alpha z^n = Q(z) - \alpha z^n
\]
has all its zeros in \( |z| < 1 \) and \( |F(z)| = |G(z)| \) for \( |z| = 1 \). Hence \( |G(z)/F(z)| \) is analytic on and inside the unit circle, and on the boundary \( |G(z)/F(z)| = 1 \). By the Maximum Modulus Principle it follows that \( |G(z)| < |F(z)| \) for \( |z| < 1 \). Replacing \( z \) by \( 1/z \) we get \( |z^n G(1/z)| < |z^n F(1/z)| \) for \( |z| > 1 \). Since \( z^n G(1/z) = F(z) \), we conclude that \( |F(z)| < |G(z)| \) for \( |z| > 1 \). Therefore, in particular
\[
|F(Re^{i\theta})| < |G(Re^{i\theta})| \quad \text{where } R > 1 \text{ and } 0 < \theta < 2\pi.
\]
This gives
\[
|P(Re^{i\theta}) - \alpha| < |Q(Re^{i\theta}) - \alpha R^n e^{i\theta}|, \quad 0 < \theta < 2\pi.
\]

Now choosing an argument of \( \alpha \) such that \( |Q(Re^{i\theta}) - \alpha R^n e^{i\theta}| = |\alpha| R^n - |Q(Re^{i\theta})| \), we obtain
\[
|P(Re^{i\theta})| - |\alpha| < |\alpha| R^n - |Q(Re^{i\theta})|, \quad 0 < \theta < 2\pi.
\]
Equivalently
\[
|P(Re^{i\theta})| + |Q(Re^{i\theta})| < |\alpha|(R^n + 1), \quad 0 < \theta < 2\pi.
\]

Letting now \( |\alpha| \to 1 \), we obtain (7) and this proves the Lemma.

Proof of Theorem 1. Since the polynomial \( P(z) \) has all its zeros in \( |z| > k > 1 \), we write
\[
P(z) = \prod_{j=1}^{n} (z - r_j e^{i\theta}) \quad \text{where } r_j > k, j = 1, 2, \ldots, n.
\]
Then \( Q(z) = z^n P(1/z) = \prod_{j=1}^{n} (1 - z r_j e^{-i \theta}) \) so that clearly we have for \( 0 < \theta < 2\pi \) and for \( R > 1 \)

\[
|P(Re^{i\theta})/Q(Re^{i\theta})| = \prod_{j=1}^{n} \left| \left( Re^{i\theta} - r_j e^{i\theta} \right) / \left( 1 - R r_j e^{i(\theta - \theta)} \right) \right|
\]

\[
= \prod_{j=1}^{n} \left| \left( Re^{i(\theta - \theta)} - r_j \right) / \left( 1 - r_j Re^{i(\theta - \theta)} \right) \right|
\]

\[
< \prod_{j=1}^{n} \left( R + r_j \right) / \left( 1 + R r_j \right)
\]

\[
< \prod_{j=1}^{n} \left( R + k \right) / \left( 1 + R k \right) = (R + k)^n / (1 + R k)^n.
\]

This implies

\[
((1 + R k)^n / (R + k)^n)|P(Re^{i\theta})| < |Q(Re^{i\theta})|, \quad 0 < \theta < 2\pi.
\]

Combining this with the conclusion of the Lemma we obtain

\[
(1 + (1 + R k)^n / (R + k)^n)|P(Re^{i\theta})| < R^n + 1, \quad 0 < \theta < 2\pi.
\]

Consequently

\[
\max_{|z| > R > 1} |P(z)| < \frac{(R^n + 1)(R + k)^n}{(R + k)^n + (1 + R k)^n},
\]

which is the desired result.

**Proof of Theorem 2.** Since all the zeros of \( P(z) \) lie in \( |z| > k > 1 \), we write as before

\[
P(z) = \prod_{j=1}^{n} \left( z - r_j e^{i\theta} \right) \quad \text{where} \quad r_j > k, j = 1, 2, \ldots, n.
\]

Then \( Q(z) = z^n P(1/z) = \prod_{j=1}^{n} (1 - z r_j e^{-i\theta}) \). Now

\[
|P(k^2 e^{i\theta})/Q(e^{i\theta})| = \prod_{j=1}^{n} \left| \left( k^2 e^{i\theta} - r_j e^{i\theta} \right) / \left( 1 - r_j e^{i(\theta - \theta)} \right) \right|
\]

\[
= \prod_{j=1}^{n} \left| \left( k^2 e^{i(\theta - \theta)} - r_j \right) / \left( 1 - r_j e^{i(\theta - \theta)} \right) \right|
\]

\[
< \prod_{j=1}^{n} k = k^n \quad \text{for} \quad 0 < \theta < 2\pi.
\]

Therefore, we have

\[
|P(k^2 z)| < k^n |Q(z)| \quad \text{for} \quad |z| = 1.
\]

(8)

Since all the zeros of \( Q(z) \) lie in \( |z| < 1/k < 1 \), it follows from the Maximum Modulus Principle that

\[
|P(k^2 z)| < k^n |Q(z)| \quad \text{for} \quad |z| > 1.
\]

Hence in particular we have

\[
|P(k^2 Re^{i\theta})| < k^n |Q(Re^{i\theta})| \quad \text{for} \quad R > 1 \quad \text{and} \quad 0 < \theta < 2\pi.
\]
This inequality with the help of the Lemma gives

$$|P(k^2Re^{i\theta})| + k^n|P(Re^{i\theta})| < k^n(R^n + 1), \quad 0 < \theta < 2\pi.$$  (9)

Now for every given $\theta$, $0 < \theta < 2\pi$, and for $R > 1$, we have either

$$|P(k^2Re^{i\theta})| - |P(Re^{i\theta})| < (k^n - 1)R^n$$  (10)

or

$$|P(k^2Re^{i\theta})| - |P(Re^{i\theta})| > (k^n - 1)R^n.$$  (11)

Inequality (10) yields with the help of inequality (9)

$$(1 + k^n)|P(k^2Re^{i\theta})| < (k^{2n}R^n + k^n)$$

and inequality (11) yields with the help of inequality (9)

$$(1 + k^n)|P(Re^{i\theta})| < |P(k^2Re^{i\theta})| + k^n|P(Re^{i\theta})| - (k^n - 1)R^n$$

$$< k^n(R^n + 1) - (k^n - 1)R^n = R^n + k^n.$$  

Thus we have either

$$\max_{|z|=R} |P(z)| < (R^n + k^n)/(1 + k^n) \quad \text{for} \quad R > k^2$$  (12)

or

$$\max_{|z|=R} |P(z)| < (R^n + k^n)/(1 + k^n) \quad \text{for} \quad R > 1.$$  (13)

From (12) and (13), it follows that, we have in any case

$$\max_{|z|=R} |P(z)| < \frac{R^n + k^n}{1 + k^n} \quad \text{for} \quad R > k^2$$

and this proves the inequality (5).

To prove the inequality (6), we observe that for $1 < R < k^2$ and $0 < \theta < 2\pi$

$$|P(Re^{i\theta})|/P(e^{i\theta})| = \prod_{j=1}^{n} \left| \frac{(Re^{i\theta} - r_je^{i\theta})}{(e^{i\theta} - r_je^{i\theta})} \right|$$

$$= \prod_{j=1}^{n} \left| \frac{(Re^{i(\theta-\theta_j)} - r_j)}{(e^{i(\theta-\theta_j)} - r_j)} \right|$$

$$< \prod_{j=1}^{n} \frac{(R + r_j)}{(1 + r_j)}$$

$$< \prod_{j=1}^{n} \frac{(R + k)}{(1 + k)} = (R + k)^n/(1 + k)^n.$$  

Consequently

$$\max_{|z|=R} |P(z)| < \frac{(R + k)^n}{(1 + k)^n} \quad \text{for} \quad 1 < R < k^2.$$  

This is the inequality (6) and the theorem is completely established.
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REFERENCES


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