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A NOTE ON THE FACTORIZATION OF OPERATOR VALUED FUNCTIONS

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ABSTRACT. Devinatz showed the factorization of positive operator valued functions $T(e^{i\theta})$ such that $\int_0^{2\pi} \log\|T(e^{i\theta})^{-1}\|^{-1}\, d\theta > -\infty$. The purpose of this note is the factorization in case $\int_0^{2\pi} \log\|T(e^{i\theta})^{-1}\|^{-1}\, d\theta = -\infty$.

1. Wiener (cf. [3, p. 122]) gave the factorization theorem of nonsingular positive matrix valued functions into analytic components. The theorem has the complement for singular positive matrix valued functions proved by Helson and Lowdenslager (cf. [3, p. 122]). Devinatz [2] extended this theorem to the situation where the ranges of functions are bounded linear operators acting on a separable Hilbert space.

Let $\mathcal{H}$ be a separable Hilbert space and let $L^2(\mathcal{H})$ denote the Hilbert space of functions defined on the unit circle with values in $\mathcal{H}$ that are weakly measurable and have square summable norm with respect to normalized Lebesgue measure $d\theta$. The $L^2(\mathcal{H})$-inner product of two functions $f = f(e^{i\theta})$ and $g = g(e^{i\theta})$ is given by $\int_0^{2\pi}(f(e^{i\theta}), g(e^{i\theta}))_{\mathcal{H}}\, d\theta$. The subspace $H^2(\mathcal{H})$ is the subset of $L^2(\mathcal{H})$ consisting of the functions $f = f(e^{i\theta})$ such that $\int_0^{2\pi}(f(e^{i\theta}), x)_{\mathcal{H}} e^{in\theta}\, d\theta = 0$ for every x in $\mathcal{H}$ when $n > 0$.

Let $\mathfrak{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$ and let $L^\infty(\mathfrak{B}(\mathcal{H}))$ denote the algebra of functions defined on the unit circle with values in $\mathfrak{B}(\mathcal{H})$ that are weakly measurable and have essentially bounded norm $\|W\|_\infty =$ ess $\sup\|W(e^{i\theta})\|_{\mathfrak{B}(\mathcal{H})}$. The subalgebra $H^\infty(\mathfrak{B}(\mathcal{H}))$ is the subset of $L^\infty(\mathfrak{B}(\mathcal{H}))$ consisting of the functions $A = A(e^{i\theta})$ such that Ax is in $H^2(\mathcal{H})$ for every x in $\mathcal{H}$. When W is a function in $L^\infty(\mathfrak{B}(\mathcal{H}))$ whose values are nonnegative almost everywhere (a.e.), W is said to be factorable if $W(e^{i\theta}) = A(e^{i\theta})^*A(e^{i\theta})$ a.e. θ for some A in $H^\infty(\mathfrak{B}(\mathcal{H}))$.

Let us now state the factorization theorem, associated with Wiener and Devinatz (cf. [3, pp. 119–123]).

WIENER-DEVINATZ THEOREM. *Suppose W is a function in $L^\infty(\mathfrak{B}(\mathcal{H}))$ whose values are nonnegative and invertible a.e. θ. If*

$$\int_0^{2\pi} \log\left\|W(e^{i\theta})^{-1}\right\|_{\mathfrak{B}(\mathcal{H})}^{-1} d\theta > -\infty,$$

then W is factorable.

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If $\dim \mathcal{K} < \infty$, *the sufficient condition for W to be factorable is equivalent to the following one*:

$$\int_0^{2\pi} \log \det\left[W(e^{i\theta}) \right] d\theta > -\infty,$$

where $[W(e^{i\theta})]$ *denotes the matrix of the operator* $W(e^{i\theta})$ *with respect to any basis in* $\mathcal{K}$ *and* $\det[W(e^{i\theta})]$ *denotes its determinant. Moreover it is the necessary condition, too.*

2. The Wiener-Devinatz Theorem is applicable only to operators W having inverses a.e. θ. But if the range of W is closed a.e. θ, some reduction may bring the problem within the scope of the theorem (cf. [3, p. 120], [1]). However when $\dim \mathcal{K} = \infty$, the sufficient condition of the theorem is strong and not necessary. We shall weaken the hypothesis and we shall give the sufficient condition of the theorem for some class of functions in $L^\infty(\mathcal{B}(\mathcal{K}))$. It follows from the theorem that when $W \in L^\infty(\mathcal{B}(\mathcal{K}))$ such that $PW(e^{i\theta})P = W(e^{i\theta}) \gg 0$ a.e. θ for some finite rank projection $P \in \mathcal{B}(\mathcal{K})$, there exists a sufficient and necessary condition for the factorability even if $\dim \mathcal{K} = \infty$. In this note, we shall prove some factorization theorem for $W \in L^\infty(\mathcal{B}(\mathcal{K}))$ such that $W(e^{i\theta}) \gg 0$ a.e. θ and $SW(e^{i\theta}) = W(e^{i\theta})S$ a.e. θ for some bilateral shift $S \in \mathcal{B}(\mathcal{K})$, when $\dim \mathcal{K} = \infty$.

Let $\mathcal{K} = L^2(\mathcal{C})$ for $\mathcal{C}$ some closed subspace of $\mathcal{K}$. Then $\mathcal{B}(\mathcal{K}) = \mathcal{B}(L^2(\mathcal{C})) \supset L^\infty(\mathcal{B}(\mathcal{C}))$ and so

$$L^\infty(\mathcal{B}(\mathcal{K})) \supset L^\infty\{L^\infty(\mathcal{B}(\mathcal{C}))\},$$

where $L^\infty\{L^\infty(\mathcal{B}(\mathcal{C}))\}$ denotes the subalgebra of $L^\infty(\mathcal{B}(\mathcal{K}))$ of functions whose rangers are in $L^\infty(\mathcal{B}(\mathcal{C}))$. We shall show the factorization theorem for functions in $L^\infty\{L^\infty(\mathcal{B}(\mathcal{C}))\}$ but not $L^\infty(\mathcal{B}(\mathcal{K}))$.

Let $\mathbf{L}^2(\mathcal{C})$ denote the Hilbert space of functions defined on the torus with values in $\mathcal{C}$ that are weakly measurable and have square summable norm with respect to normalized Lebesgue measure $d\theta\, d\phi$. The $\mathbf{L}^2(\mathcal{C})$-inner product of two functions $f = f(e^{i\theta}, e^{i\phi})$ and $g = g(e^{i\theta}, e^{i\phi})$ is given by $\int_0^{2\pi} \int_0^{2\pi} (f, g)_{\mathcal{C}}\, d\theta\, d\phi$. The subspace $\mathbf{H}^2(\mathcal{C})$ is the subset of $\mathbf{L}^2(\mathcal{C})$ consisting of the functions $f = f(e^{i\theta}, e^{i\theta})$ such that

$$\int_0^{2\pi} \int_0^{2\pi} (f(e^{i\theta}, e^{i\phi}), x)_{\mathcal{C}}\, e^{in\theta}\, d\theta\, d\phi = 0$$

for every x in $\mathcal{C}$ when $n > 0$. Let $\mathbf{L}^\infty(\mathcal{B}(\mathcal{C}))$ denote the algebra of functions defined on the torus with values in $\mathcal{B}(\mathcal{C})$ that are weakly measurable and have essentially bounded norm $\|T\|'_\infty = \operatorname{ess\,sup} \|T(e^{i\theta}, e^{i\phi})\|_{\mathcal{B}(\mathcal{C})}$. The subalgebra $\mathbf{H}^\infty(\mathcal{B}(\mathcal{C}))$ is the subset of $\mathbf{L}^\infty(\mathcal{B}(\mathcal{C}))$ consisting of the functions $B = B(e^{i\theta}, e^{i\phi})$ such that Bx is in $\mathbf{H}^2(\mathcal{C})$ for every x in $\mathcal{C}$.

The following lemma implies that if W is in $L^\infty\{L^\infty(\mathcal{B}(\mathcal{C}))\}$, then there exists T in $\mathbf{L}^\infty(\mathcal{B}(\mathcal{C}))$ which is unitarily equivalent to W. $I_{\mathcal{K}}$ (resp. $I_{\mathcal{C}}$) denotes an identity operator in $\mathcal{B}(\mathcal{K})$ (resp. $\mathcal{B}(\mathcal{C})$). The proof is surely known, but we give a proof for completeness.

LEMMA 1. (1) *There is a unitary operator* U *of* $L^2\{L^2(\mathcal{C})\}$ *onto* $\mathbf{L}^2(\mathcal{C})$ *with* $UH^2\{L^2(\mathcal{C})\} = \mathbf{H}^2(\mathcal{C})$.

(2) *If* W *is in* $L^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$ *and* $\Phi W = UWU^*$ *for the unitary operator* U *above, then* Φ *is an isometric isomorphism from* $L^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$ *onto* $\mathbf{L}^\infty(\mathfrak{B}(\mathcal{C}))$ *with* $\Phi H^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\} = \mathbf{H}^\infty(\mathfrak{B}(\mathcal{C}))$.

(3) *If* $W_1(e^{i\theta}) = e^{i\theta}I_{\mathcal{K}}$ *and* $W_2(e^{i\theta}) = \ldots \oplus e^{i\phi}I_{\mathcal{C}} \oplus e^{i\phi}I_{\mathcal{C}} \oplus \ldots,$ *then* $(\Phi W_1)(e^{i\theta}, e^{i\phi}) = e^{i\theta}I_{\mathcal{C}}$ *and* $(\Phi W_2)(e^{i\theta}, e^{i\phi}) = e^{i\phi}I_{\mathcal{C}}$.

PROOF. Let L^2 and L^∞ be the usual Lebesgue spaces on the unit circle, then a unitary operator U in (1) is given by $U(f_1 \otimes f_2 \otimes e) = f_1(e^{i\theta})f_2(e^{i\phi})e$ for $f_1, f_2 \in L^2 \cap L^\infty$ and $e \in \mathcal{C}$, where '$\otimes$' denotes the tensor product. If $\Phi(g_1 \otimes g_2 \otimes T_0) = g_1(e^{i\theta})g_2(e^{i\phi})T_0$ for $g_1, g_2 \in L^\infty$ and $T_0 \in \mathfrak{B}(\mathcal{C})$, then $\Phi W = UWU^*$ for $W \in L^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$ showing (2) and (3).

When dim $\mathcal{C} = 1$, we write $\mathbf{L}^2 = \mathbf{L}^2(\mathcal{C})$ and $\mathbf{H}^2 = \mathbf{H}^2(\mathcal{C})$, and $\mathbf{L}^\infty = \mathbf{L}^\infty(\mathfrak{B}(\mathcal{C}))$ and $\mathbf{H}^\infty = \mathbf{H}^\infty(\mathfrak{B}(\mathcal{C}))$. Then $\mathbf{H}^\infty$ is the weak-*closure of polynomials of $e^{i\theta}$, $e^{i\phi}$, $e^{-i\phi}$ with respect to $d\theta\, d\phi$ on the torus and $\mathbf{H}^2$ is the norm closure of $\mathbf{H}^\infty$. The following lemma is a scalar theorem for our factorization theorem.

LEMMA 2. *Suppose* $t = t(e^{i\theta}, e^{i\phi})$ *is a positive function in* $\mathbf{L}^\infty$. *Then*

$$\int_0^{2\pi} \log t(e^{i\theta}, e^{i\phi})d\theta > -\infty \quad a.e.\ \phi$$

if and only if $t = |g|^2$ *for some* g *in* $\mathbf{H}^\infty$.

PROOF. If $g \in \mathbf{H}^\infty$ and $|g(e^{i\theta}, e^{i\phi})| > 0$ a.e. (θ, ϕ), then $g(e^{i\theta}, e^{i\phi})$ is a nonzero function in the classical Hardy space H^∞ for almost every ϕ. Hence

$$\int_0^{2\pi} \log|(e^{i\theta}, e^{i\phi})|d\theta > -\infty \quad \text{a.e. } \phi$$

[3, p. 21].

If $t \in \mathbf{L}^\infty$, $t(e^{i\theta}, e^{i\phi}) > 0$ a.e. (θ, ϕ) and $\int_0^{2\pi} \log t(e^{i\theta}, e^{i\phi})d\theta > -\infty$ a.e. ϕ, then

$$\inf_p \int_0^{2\pi}\left|1 - p(e^{i\theta}, e^{i\phi})\right|^2 t(e^{i\theta}, e^{i\phi})d\theta > -\infty \quad \text{a.e. } \phi \tag{$*$}$$

where p ranges over polynomials of $e^{i\theta}$, $e^{i\phi}$ and $e^{-i\phi}$ with mean value zero by Szegö's theorem [3, p. 19]. Let $\mathfrak{M}_{t^{1/2}}$ be the closed linear span of the functions $ft^{1/2}$, $f \in \mathbf{H}^\infty$ in $\mathbf{L}^2$. If $f \in \mathfrak{M}_{t^{1/2}}$ is orthogonal to $e^{i\theta}\mathfrak{M}_{t^{1/2}}$, then for almost everywhere ϕ $|f(e^{i\theta}, e^{i\phi})| = 1$ a.e. θ by ($*$) and Beurling's theorem [3, p. 8]. Thus $\mathfrak{M}_{t^{1/2}} \supseteq f\mathbf{H}^2$ and $\mathfrak{M}_{t^{1/2}} = f(e^{i\theta}, e^{i\phi})H^2$ a.e. ϕ by Beurling's theorem, so $\mathfrak{M}_{t^{1/2}} = f\mathbf{H}^2$, where H^2 is the classical Hardy space. This implies $t = |g|^2$ for some $g \in \mathbf{H}^\infty$.

Suppose W is a function in $L^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$ whose values are nonnegative a.e. θ and Φ is an isometric isomorphism from $L^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$ onto $\mathbf{L}^\infty(\mathfrak{B}(\mathcal{C}))$ in Lemma 1. Then, for a.e. θ

$$\inf\{(W(e^{i\theta})f, f)_{\mathcal{K}}; f \in \mathcal{K} \text{ and } \|f\| = 1\}$$
$$< \Phi^{-1}\inf\{((\Phi W)(e^{i\theta}, e^{i\phi})x, x)_{\mathcal{C}}; x \in \mathcal{C} \text{ and } \|x\| = 1\}$$

and so if W is invertible a.e. θ, then ΦW is invertible a.e. (θ, ϕ). The converse is not valid. If $W(e^{i\theta})^{-1} \in \mathfrak{B}(\mathcal{K})$ a.e. θ, for a.e. θ

$$\left\| W(e^{i\theta})^{-1} \right\|_{\mathfrak{B}(\mathcal{K})}^{-1} \leqslant \Phi^{-1} \left\| (\Phi W)(e^{i\theta}, e^{i\phi})^{-1} \right\|_{\mathfrak{B}(\mathcal{C})}^{-1}$$

and

$$\int_0^{2\pi} \log \left\| W(e^{i\theta})^{-1} \right\|_{\mathfrak{B}(\mathcal{K})}^{-1} d\theta = \int_0^{2\pi} \log \Phi \left\| W(e^{i\theta})^{-1} \right\|_{\mathfrak{B}(\mathcal{K})}^{-1} d\theta.$$

Now we shall show the factorization theorem for functions in $L^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$.

THEOREM. *Suppose W is a function in $L^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$ $(\subset L^\infty(\mathfrak{B}(\mathcal{K})))$ whose values are nonnegative a.e. θ. Suppose $(\Phi W)(e^{i\theta}, e^{i\phi})$ is invertible a.e. (θ, ϕ). If*

$$\int_0^{2\pi} \log \left\| (\Phi W)(e^{i\theta}, e^{i\phi})^{-1} \right\|_{\mathfrak{B}(\mathcal{C})}^{-1} d\theta > -\infty \quad \textit{a.e. } \phi,$$

then W is factorable.

If $\dim \mathcal{C} < \infty$, *the sufficient condition for W to be factorable is equivalent to the following one;*

$$\int_0^{2\pi} \log \det\left[(\Phi W)(e^{i\theta}, e^{i\phi})\right] d\theta > -\infty \quad \textit{a.e. } \phi,$$

where $[(\Phi W)(e^{i\theta}, e^{i\phi})]$ denotes the matrix of the operator $(\Phi W)(e^{i\theta}, e^{i\phi})$ with respect to any basis in $\mathcal{C}$. Moreover it is the necessary condition if W is factorable with analytic components in $H^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$.

PROOF. Set $t(e^{i\theta}, e^{i\phi}) = \|(\Phi W)(e^{i\theta}, e^{i\phi})^{-1}\|_{\mathfrak{B}(\mathcal{C})}^{-1}$, then

$$(\Phi W)(e^{i\theta}, e^{i\phi}) \geqslant t(e^{i\theta}, e^{i\phi}) I_{\mathcal{C}} \quad \text{a.e. } (\theta, \phi).$$

Since $\int_0^{2\pi} \log t(e^{i\theta}, e^{i\phi}) d\theta > -\infty$ a.e. ϕ, $tI_{\mathcal{C}} = (gI_{\mathcal{C}})^*(gI_{\mathcal{C}})$, where g is a function in $\mathbf{H}^\infty$ such that $t = |g|^2$ by Lemma 2. So $\Phi^{-1}(tI_{\mathcal{C}})$ is factorable and has dense range, and $W \geqslant \Phi^{-1}(tI_{\mathcal{C}})$. Since $(\Phi W)(tI_{\mathcal{C}}) = (tI_{\mathcal{C}})(\Phi W)$, $W\{\Phi^{-1}(tI_{\mathcal{C}})\} = \{\Phi^{-1}(tI_{\mathcal{C}})\} W$ and so W is factorable by **[1]**.

When $\dim \mathcal{C} < \infty$, it is similar to [3, p. 123] that the sufficient condition is equivalent to the condition of determinant of $[\Phi W]$. If $W = G^*G$ for some $G \in H^\infty\{L^\infty(\mathfrak{B}(\mathcal{C}))\}$, then $\Phi W = (\Phi G)^*(\Phi G)$ and $\Phi G \in \mathbf{H}^\infty(\mathfrak{B}(\mathcal{C}))$. Since $\det[\Phi W] = |\det[\Phi G]|^2$ and the entries in $[\Phi G]$ are functions from $\mathbf{H}^\infty$, Lemma 2 implies

$$\int_0^{2\pi} \log \det\left[(\Phi W)(e^{i\theta}, e^{i\phi})\right] d\theta > -\infty \quad \text{a.e. } \phi.$$

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