ON UNIVERSAL NULL SETS

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Abstract. If all subsets of cardinality less than $2^K$ of the real line $R$ are Lebesgue measurable then there exists a permutation $p$ of $R$ with $p = p^{-1}$ such that on the $\sigma$-field generated by $\mathcal{B} \cup p(\mathcal{B})$ there is no continuous probability measure.

1. Let $|S|$ denote the cardinality of the set $S$. If $f$ is a function from a set $S$ into a set $T$ and $\mathcal{F}$ is a family of subsets of $S$ then by $f(\mathcal{F})$ we denote the family $\{f(F) : F \in \mathcal{F}\}$ of subsets of $T$. Let $\mathcal{G}$ be a $\sigma$-field on $S$ (i.e. a countably additive algebra of subsets of $S$). If $X \subset S$ then $\mathcal{G} \cap X$ will denote the $\sigma$-field $\{C \cap X : C \in \mathcal{G}\}$ on $X$. A countably additive measure $\mu$ on a countably generated $\sigma$-field $\mathcal{G}$ on $S$ will be called a nontrivial continuous measure iff $0 < \mu(S) < \infty$ and $\mu$ vanishes for all atoms of $\mathcal{G}$. We will say that $\mathcal{G}$ is measurable if there exists a nontrivial continuous measure on $\mathcal{G}$. Otherwise one says that $\mathcal{G}$ is nonmeasurable (see [4]).

Recall that a subset $Y$ of a separable metrizable space $X$ is called a universal null set iff for every nontrivial continuous measure $\mu$ on $\mathcal{B}_X$ (= Borel subsets of $X$) we have $\mu^*(Y) = 0$, where $\mu^*$ is the outer measure induced by $\mu$ (see [1], [7] and [13]).

It is easy to check that $Y$ is a universal null set iff $\mathcal{B}_Y$ is a nonmeasurable $\sigma$-field on $Y$ (see [11] and [13]). Marczewski and Sierpiński discovered in [11] an uncountable universal null subset of the real line. Further information on universal null sets can be found e.g. in [1], [2], [4], [6], [7], [11] and [13], where there are also other references. A separable metric space $X$ is called absolute Borel if $X$ is a Borel subset of its completion.

2. We work in ZFC set theory.

If $X$ is a separable metrizable space, then we shall consider the following conditions concerning $X$.

(i) There exists a universal null subset $Y$ of $X$ with $|Y| = |X|$.

(ii) There exists a permutation $p$ of $X$ with $p = p^{-1}$ and such that the graph of $p$ is a universal null subset of $X \times X$.

(iii) There exists a permutation $p$ of $X$ with $p = p^{-1}$ and such that the $\sigma$-field generated by $\mathcal{B}_X \cup p(\mathcal{B}_X)$ is nonmeasurable.

In connection with (i) it is worth mentioning that for $X = R$ condition (i) is a theorem of ZFC + all subsets of $R$ of cardinality less than $2^K$ are Lebesgue measurable.
measurable [4], but (i) is independent of ZFC + $2^\omega > \aleph_1$. More precisely, Laver [6, p. 152] has announced the existence of a model for ZFC + $2^\omega > \aleph_1$ in which there are no universal null subsets of $R$ of power greater than $\aleph_1$. It is known that Martin's Axiom (MA) implies that all subsets of $R$ of cardinality $< 2^\omega$ are Lebesgue measurable; moreover ZFC + MA + $2^\omega > \aleph_1$ is consistent provided ZFC is consistent. More references can be found in [6] and [8].

The aim of this note is to prove the following

**Theorem.** (a) (ii) if and only if (iii).
(b) (i) implies (iii).
(c) If $X$ contains an uncountable absolute Borel subset then (ii) implies (i).
(d) There exists a set $X \subset R$ of positive outer Lebesgue measure such that (i) holds for $X$.

The following corollaries are strengthenings of Theorems 1 and 2 from [4].

**Corollary 1.** There exist a countably generated and separating points measurable $\sigma$-field $G$ on a set $S$ and a permutation $p$ of $S$ with $p = p^{-1}$ and such that the $\sigma$-field generated by $G \cup p(G)$ is nonmeasurable.

**Corollary 2.** There exist a countably generated measurable $\sigma$-field $G$ on $R$ and a permutation $p$ of $R$ with $p = p^{-1}$ and such that the $\sigma$-field generated by $G \cup p(G)$ is nonmeasurable.

We do not know the answer to the following questions.

(1) Let $X$ be a nonempty subset of $R$ such that each subset $Y$ of $X$ with $|Y| = |X|$ has positive outer Lebesgue measure (the existence of such sets has been proved by Sierpiński (see e.g. [10]) under hypothesis $2^\omega = \aleph_1$, in fact MA is sufficient). It is evident that all such $X$'s do not have property (i). Does $X$ have property (iii)?

(2) Does there exist a subset $X$ of $R$ such that (iii) holds but (i) fails?

3. We start with a proof of the theorem. Part (a) follows easily from the following

**Lemma.** Let $C$ be a $\sigma$-field on a set $S$, let $p$ be a permutation of $S$ and let $P \subset S \times S$ be the graph of $p$. Then the $\sigma$-field $\mathcal{C}^P$ on $S$ generated by $C \cup p(C)$ is point isomorphic with the $\sigma$-field $(C \otimes C) \cap P$, where $C \otimes C$ is the product $\sigma$-field on $S \times S$.

**Proof of the Lemma.** Consider the projection map $\pi: P \to S$, $\pi(x, y) = y$ where $(x, y) \in P$. Since $p$ is a permutation, $\pi$ is a one-to-one map from $P$ onto $S$. For every $C \in C$ we have $\pi((C \times S) \cap P) = p(C)$ and $\pi((S \times C) \cap P) = C$; consequently $\mathcal{C}^P = \pi((C \otimes C) \cap P)$. So $\pi$ is an isomorphism between $(C \otimes C) \cap P$ and $\mathcal{C}^P$.

Let $p$ be a permutation of a separable metrizable space $X$ and let $P \subset X \times X$ be the graph of $p$. Since $(\mathcal{B}_X \otimes \mathcal{B}_X) \cap P = \mathcal{B}_P$, by the lemma we have that the $\sigma$-field $\mathcal{B}_P$ is nonmeasurable if and only if the $\sigma$-field generated by $\mathcal{B}_X \cup p(\mathcal{B}_X)$ is nonmeasurable. Hence conditions (ii) and (iii) are mutually equivalent.
Proof of (b). Assume that $X$ satisfies (i). For finite $X$ (iii) is obvious; assume that $X$ is infinite. Hence there exists a universal null subset $Y$ of $X$ with $|Y| = |X \setminus Y|$. Let $p$ be a permutation of $X$ such that $p = p^{-1}$ and $p(Y) = X \setminus Y$. We claim that $p$ satisfies (iii). If not, let $\mu$ be a nontrivial continuous measure on the $\sigma$-field generated by $B_X \cup p(B_X)$. Let $\mu_1$ be the restriction of $\mu$ to $B_X$ and let $\mu_2$ be the restriction of $\mu$ to the $\sigma$-field $p(B_X)$. Define a nontrivial continuous measure $\nu$ on $B_X$ putting $\nu(B) = \mu_2(p(B))$ for all $B \in B_X$. Since $Y$ is a universal null subset of $X$ we have $\mu_1^*(Y) = \mu_2^*(Y) = 0$; moreover $\nu^*(Y) = \mu_2^*(p(Y)) = \mu_2^*(X \setminus Y)$. So

$$\mu_1^*(X) < \mu_1^*(Y) + \mu_2^*(X \setminus Y) < \mu_1^*(Y) + \mu_2^*(X \setminus Y) = 0;$$

hence we get a contradiction.

Proof of (c). Assume that $X$ satisfies (ii). Applying Marczewski's theorem (see [12, p. 144]) we can obtain a subset $Z$ of $R$ such that the $\sigma$-field $B_{X \times X} \cap (\text{graph } p)$ is isomorphic with the $\sigma$-field $B_Z$. Let $B$ be an uncountable absolute Borel subset of $X$. It is well known that $B_{B \times B}$ is isomorphic to $B_R$ (see e.g. [5, p. 451]). Hence there exists a subset $Y$ of $B$ such that $B_{X \times X} \cap (\text{graph } p)$ is isomorphic with $B_Y$. Since graph $p$ is a universal null subset of $X \times X$ we have that $\nu(x) = \mu_2^*(p(B)) = \mu_2^*(X \setminus Y)$. Therefore $Y$ is a universal null subset of $X$ and $|Y| = |\text{graph } p| = |X|$.

Proof of (d). There exist (see Corollary 2 in [4]) subsets $A$ and $B$ of $R$ such that $|A| = |B|$. $A$ is a universal null set (called in [4] absolutely of measure zero) and $B$ has positive outer Lebesgue measure. Put $X = A \cup B$. It is evident that $X$ has property (i).

Proof of Corollary 1. Let $X$ be as in the theorem, part (d). By part (b) there exists a permutation $p$ of $X$ satisfying (iii). Put $C = B_X$ and $S = X$. Since outer Lebesgue measure restricted to $B_X$ is a nonmeasurable $\sigma$-field on $B_X$ we have that $C$ is a measurable $\sigma$-field as was required.

Proof of Corollary 2. Let $X$ and $p$ be as in the theorem, part (d). Let $C$ be the $\sigma$-field on $R$ generated by $B_X \cup \{R \setminus X\}$. Since, as it was observed in the proof of Corollary 1, $B_X$ is measurable we have that $C$ is a countably generated measurable $\sigma$-field on $R$. Let $p'$ be a permutation of $R$ defined by $p'(x) = p(x)$ for all $x \in X$, and $p'(x) = x$ for all $x \in R \setminus X$. It is easy to check that $p'$ and $C$ have the required properties.

4. If $X$ and $Y$ are separable metrizable spaces such that $|X| = |Y|$, then we shall consider the following conditions concerning $X$ and $Y$.

(i') There exist universal null subsets $X_1$ of $X$ and $Y_1$ of $Y$ with $|X_1| = |X|$ and $|Y_1| = |Y|$.

(ii') There exists a one-to-one function $p$ from $X$ onto $Y$ such that the graph of $p$ is a universal null subset of $X \times Y$.

(iii') There exists a one-to-one function $p$ from $X$ onto $Y$ such that the $\sigma$-field generated by $B_Y \cup p(B_X)$ is nonmeasurable.

Recall that a subset $X_1$ of a separable metrizable space $X$ is called universally measurable iff for every probability measure $\mu$ on $B_X$ there exist $B_1, B_2 \in B$ such that $B_1 \subset X \subset B_2$ and $\mu(B_2 \setminus B_1) = 0$ (see e.g. [7]).
A proof of the following Theorem' is an easy modification of the proof of the theorem, and will be omitted.

**Theorem.** (a) (ii') if and only if (iii').
(b) (i') implies (iii').
(c) If \(X\) and \(Y\) contain uncountable absolute Borel sets then (ii') implies (i').
(d) If \(B\) is an uncountable absolute Borel set then there exists a subset \(X\) of \(B\) such that \(X\) is not universally measurable and there exists a universal null subset \(X_1\) of \(X\) with \(|X_1| = |X|\).

The following Corollary 3 will be useful.

**Corollary 3.** Let \(K\) and \(C\) be uncountable absolute Borel sets. Then there exists a universal null subset \(N\) of \(K \times C\) such that the projections of \(N\) onto \(K\) and onto \(C\) are not universally measurable subsets of \(K\) and \(C\), respectively.

**Proof of Corollary 3.** By part (d) of Theorem' there exists a subset \(X\) of \(K\) such that \(X\) is not universally measurable and there exist a universal null subset \(X_1\) of \(X\) with \(|X_1| = |X|\). Let \(f\) be a Borel isomorphism between \(K\) and \(C\). Put \(Y = f(X)\) and \(Y_1 = f(X_1)\). It is easy to see that \(Y\) is a subset of \(C\) which is not universally measurable, \(Y_1\) is a universal null subset of \(Y\) and \(|Y_1| = |Y| = |X|\). By Theorem' we have that (i') implies (ii'). Hence there exists a one-to-one function \(p\) from \(X\) onto \(Y\) such that the graph of \(p\), \(N\), is a universal null subset of \(X \times Y\). It is evident that \(N\) has the properties required in Corollary 3.

Corollary 3 permits the elimination of MA from the assumption of the theorem of Darst [3], that a Borel function \(f\), mapping a Borel subset, \(D_f\), of a separable complete metric space, \(M_1\), into a separable complete metric space, \(M_2\), maps Borel subsets of \(D_f\) onto Borel subsets of \(M_2\) if and only if \(f\) maps universally measurable subsets of \(D_f\) onto universally measurable subsets of \(M_2\).

Darst has proved his theorem with the help of the continuum hypothesis and has written [3, p. 566]: "We shall need to employ the continuum hypothesis only in the last step of our argument."

The mentioned last step of the proof of Darst is a part of Corollary 3. Mauldin has proved that the continuum hypothesis can be replaced there by MA [9, p. 60–61].

**Remark 1.** The implication that (i') imples (ii') can also be proved easily directly.

**Remark 2 (Added March 15, 1980).** In 1979 G. V. Cox (*A universal null graph whose domain has positive measure*, Colloq. Math. (to appear)), assuming the continuum hypothesis, answered both our questions. Namely he proved, in ZFC + CH, the existence of two uncountable subsets \(X_1\), \(X_2\) of \(R\) such that each uncountable subset of \(X_1\) and each uncountable subset of \(X_2\) have positive outer Lebesgue measure, (ii) holds for \(X_1\) and (ii) does not hold for \(X_2\).

**References**


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