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ON UNIVERSAL NULL SETS

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ABSTRACT. If all subsets of cardinality less than $2^{\aleph_0}$ of the real line R are Lebesgue measurable then there exists a permutation p of R with $p = p^{-1}$ such that on the σ-field generated by $\mathcal{B} \cup p(\mathcal{B})$ there is no continuous probability measure.

1. Let $|S|$ denote the cardinality of the set S. If f is a function from a set S into a set T and $\mathcal{F}$ is a family of subsets of S then by $f(\mathcal{F})$ we denote the family $\{f(F): F \in \mathcal{F}\}$ of subsets of T. Let $\mathcal{C}$ be a σ-field on S (i.e. a countably additive algebra of subsets of S). If $X \subset S$ then $\mathcal{C} \cap X$ will denote the σ-field $\{C \cap X: C \in \mathcal{C}\}$ on X. A countably additive measure μ on a countably generated σ-field $\mathcal{C}$ on S will be called a nontrivial continuous measure iff $0 < \mu(S) < \infty$ and μ vanishes for all atoms of $\mathcal{C}$. We will say that $\mathcal{C}$ is *measurable* if there exists a nontrivial continuous measure on $\mathcal{C}$. Otherwise one says that $\mathcal{C}$ is *nonmeasurable* (see **[4]**).

Recall that a subset Y of a separable metrizable space X is called a *universal null* set iff for every nontrivial continuous measure μ on $\mathcal{B}_X$ (= Borel subsets of X) we have $\mu^*(Y) = 0$, where μ^* is the outer measure induced by μ (see **[1]**, **[7]** and **[13]**).

It is easy to check that Y is a universal null set iff $\mathcal{B}_Y$ is a nonmeasurable σ-field on Y (see **[11]** and **[13]**). Marczewski and Sierpiński discovered in **[11]** an uncountable universal null subset of the real line. Further information on universal null sets can be found e.g. in **[1]**, **[2]**, **[4]**, **[6]**, **[7]**, **[11]** and **[13]**, where there are also other references. A separable metric space X is called absolute Borel if X is a Borel subset of its completion.

2. We work in ZFC set theory.

If X is a separable metrizable space, then we shall consider the following conditions concerning X.

(i) There exists a universal null subset Y of X with $|Y| = |X|$.

(ii) There exists a permutation p of X with $p = p^{-1}$ and such that the graph of p is a universal null subset of $X \times X$.

(iii) There exists a permutation p of X with $p = p^{-1}$ and such that the σ-field generated by $\mathcal{B}_X \cup p(\mathcal{B}_X)$ is nonmeasurable.

In connection with (i) it is worth mentioning that for $X = R$ condition (i) is a theorem of ZFC + all subsets of R of cardinality less than $2^{\aleph_0}$ are Lebesgue

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measurable [4], but (i) is independent of ZFC + $2^{\aleph_0} > \aleph_1$. More precisely, Laver [6, p. 152] has announced the existence of a model for ZFC + $2^{\aleph_0} > \aleph_1$ in which there are no universal null subsets of R of power greater than $\aleph_1$. It is known that Martin's Axiom (MA) implies that all subsets of R of cardinality $< 2^{\aleph_0}$ are Lebesgue measurable; moreover ZFC + MA + $2^{\aleph_0} > \aleph_1$ is consistent provided ZFC is consistent. More references can be found in [6] and [8].

The aim of this note is to prove the following

THEOREM. (a) (ii) *if and only if* (iii).

(b) (i) *implies* (iii).

(c) *If X contains an uncountable absolute Borel subset then* (ii) *implies* (i).

(d) *There exists a set $X \subset R$ of positive outer Lebesgue measure such that* (i) *holds for X.*

The following corollaries are strengthenings of Theorems 1 and 2 from [4].

COROLLARY 1. *There exist a countably generated and separating points measurable σ-field $\mathcal{C}$ on a set S and a permutation p of S with $p = p^{-1}$ and such that the σ-field generated by $\mathcal{C} \cup p(\mathcal{C})$ is nonmeasurable.*

COROLLARY 2. *There exist a countably generated measurable σ-field $\mathcal{C}$ on R and a permutation p of R with $p = p^{-1}$ and such that the σ-field generated by $\mathcal{C} \cup p(\mathcal{C})$ is nonmeasurable.*

We do not know the answer to the following questions.

(1) Let X be a nonempty subset of R such that each subset Y of X with $|Y| = |X|$ has positive outer Lebesgue measure (the existence of such sets has been proved by Sierpiński (see e.g. [10]) under hypothesis $2^{\aleph_0} = \aleph_1$, in fact MA is sufficient). It is evident that all such X's do not have property (i). Does X have property (iii)?

(2) Does there exist a subset X of R such that (iii) holds but (i) fails?

3. We start with a proof of the theorem. Part (a) follows easily from the following

LEMMA. *Let $\mathcal{C}$ be a σ-field on a set S, let p be a permutation of S and let $P \subset S \times S$ be the graph of p. Then the σ-field $\mathfrak{D}$ on S generated by $\mathcal{C} \cup p(\mathcal{C})$ is point isomorphic with the σ-field $(\mathcal{C} \otimes \mathcal{C}) \cap P$, where $\mathcal{C} \otimes \mathcal{C}$ is the product σ-field on $S \times S$.*

PROOF OF THE LEMMA. Consider the projection map π: $P \to S$, $\pi(x, y) = y$ where $(x, y) \in P$. Since p is a permutation, π is a one-to-one map from P onto S. For every $C \in \mathcal{C}$ we have $\pi((C \times S) \cap P) = p(C)$ and $\pi((S \times C) \cap P) = C$; consequently $\mathfrak{D} = \pi((\mathcal{C} \otimes \mathcal{C}) \cap P)$. So π is an isomorphism between $(\mathcal{C} \otimes \mathcal{C}) \cap P$ and $\mathfrak{D}$.

Let p be a permutation of a separable metrizable space X and let $P \subset X \times X$ be the graph of p. Since $(\mathfrak{B}_X \otimes \mathfrak{B}_X) \cap P = \mathfrak{B}_P$, by the lemma we have that the σ-field $\mathfrak{B}_P$ is nonmeasurable if and only if the σ-field generated by $\mathfrak{B}_X \cup p(\mathfrak{B}_X)$ is nonmeasurable. Hence conditions (ii) and (iii) are mutually equivalent.

PROOF OF (b). Assume that X satisfies (i). For finite X (iii) is obvious; assume that X is infinite. Hence there exists a universal null subset Y of X with $|Y| = |X \setminus Y|$. Let p be a permutation of X such that $p = p^{-1}$ and $p(Y) = X \setminus Y$. We claim that p satisfies (iii). If not, let μ be a nontrivial continuous measure on the σ-field generated by $\mathfrak{B}_X \cup p(\mathfrak{B}_X)$. Let μ_1 be the restriction of μ to $\mathfrak{B}_X$ and let μ_2 be the restriction of μ to the σ-field $p(\mathfrak{B}_X)$. Define a nontrivial continuous measure ν on $\mathfrak{B}_X$ putting $\nu(B) = \mu_2(p(B))$ for all $B \in \mathfrak{B}_X$. Since Y is a universal null subset of X we have $\mu_1^*(Y) = \nu^*(Y) = 0$; moreover $\nu^*(Y) = \mu_2^*(p(Y)) = \mu_2^*(X \setminus Y)$. So

$$\mu^*(X) < \mu^*(Y) + \mu^*(X \setminus Y) < \mu_1^*(Y) + \mu_2^*(X \setminus Y) = 0;$$

hence we get a contradiction.

PROOF OF (c). Assume that X satisfies (ii). Applying Marczewski's theorem (see [12, p. 144]) we can obtain a subset Z of R such that the σ-field $\mathfrak{B}_{X \times X} \cap (\text{graph}\, p)$ is isomorphic with the σ-field $\mathfrak{B}_Z$. Let B be an uncountable absolute Borel subset of X. It is well known that $\mathfrak{B}_B$ is isomorphic to $\mathfrak{B}_R$ (see e.g. [5, p. 451]). Hence there exists a subset Y of B such that $\mathfrak{B}_{X \times X} \cap (\text{graph}\, p)$ is isomorphic with $\mathfrak{B}_Y$. Since graph p is a universal null subset of $X \times X$ we have that $(\text{graph}\, p) \cap \mathfrak{B}_{X \times X}$ is a nonmeasurable σ-field and hence $\mathfrak{B}_Y$ is a nonmeasurable σ-field on Y. Therefore Y is a universal null subset of X and $|Y| = |\text{graph}\, p| = |X|$.

PROOF OF (d). There exist (see Corollary 2 in [4]) subsets A and B of R such that $|A| = |B|$, A is a universal null set (called in [4] absolutely of measure zero) and B has positive outer Lebesgue measure. Put $X = A \cup B$. It is evident that X has property (i).

PROOF OF COROLLARY 1. Let X be as in the theorem, part (d). By part (b) there exists a permutation p of X satisfying (iii). Put $\mathcal{C} = \mathfrak{B}_X$ and $S = X$. Since outer Lebesgue measure restricted to $\mathfrak{B}_X$ is a nontrivial continuous measure on $\mathfrak{B}_X$ we have that $\mathcal{C}$ is a measurable σ-field as was required.

PROOF OF COROLLARY 2. Let X and p be as in the theorem, part (d). Let $\mathcal{C}$ be the σ-field on R generated by $\mathfrak{B}_X \cup \{R \setminus X\}$. Since, as it was observed in the proof of Corollary 1, $\mathfrak{B}_X$ is measurable we have that $\mathcal{C}$ is a countably generated measurable σ-field on R. Let p' be a permutation of R defined by $p'(x) = p(x)$ for all $x \in X$, and $p'(x) = x$ for all $x \in R \setminus X$. It is easy to check that p' and $\mathcal{C}$ have the required properties.

4. If X and Y are separable metrizable spaces such that $|X| = |Y|$, then we shall consider the following conditions concerning X and Y.

(i′) There exist universal null subsets X_1 of X and Y_1 of Y with $|X_1| = |X|$ and $|Y_1| = |Y|$.

(ii′) There exists a one-to-one function p from X onto Y such that the graph of p is a universal null subset of $X \times Y$.

(iii′) There exists a one-to-one function p from X onto Y such that the σ-field generated by $\mathfrak{B}_Y \cup p(B_X)$ is nonmeasurable.

Recall that a subset X_1 of a separable metrizable space X is called *universally measurable* iff for every probability measure μ on $\mathfrak{B}_X$ there exist $B_1, B_2 \in \mathfrak{B}$ such that $B_1 \subset X \subset B_2$ and $\mu(B_2 \setminus B_1) = 0$ (see e.g. [7]).

A proof of the following Theorem′ is an easy modification of the proof of the theorem, and will be omitted.

THEOREM′. (a) (ii′) *if and only if* (iii′).

(b) (i′) *implies* (iii′).

(c) *If X and Y contain uncountable absolute Borel sets then* (ii′) *implies* (i′).

(d) *If B is an uncountable absolute Borel set then there exists a subset X of B such that X is not universally measurable and there exists a universal null subset X_1 of X with $|X_1| = |X|$.*

The following Corollary 3 will be useful.

COROLLARY 3. *Let K and C be uncountable absolute Borel sets. Then there exists a universal null subset N of $K \times C$ such that the projections of N onto K and onto C are not universally measurable subsets of K and C, respectively.*

PROOF OF COROLLARY 3. By part (d) of Theorem′ there exists a subset X of K such that X is not universally measurable and there exist a universal null subset X_1 of X with $|X_1| = |X|$. Let f be a Borel isomorphism between K and C. Put $Y = f(X)$ and $Y_1 = f(X_1)$. It is easy to see that Y is a subset of C which is not universally measurable, Y_1 is a universal null subset of Y and $|Y_1| = |Y| = |X|$. By Theorem′ we have that (i′) implies (ii′). Hence there exists a one-to-one function p from X onto Y such that the graph of p, N, is a universal null subset of $X \times Y$. It is evident that N has the properties required in Corollary 3.

Corollary 3 permits the elimination of MA from the assumption of the theorem of Darst [3], that *a Borel function f, mapping a Borel subset, D_f, of a separable complete metric space, M_1, into a separable complete metric space, M_2, maps Borel subsets of D_f onto Borel subsets of M_2 if and only if f maps universally measurable subsets of D_f onto universally measurable subsets of M_2.*

Darst has proved his theorem with the help of the continuum hypothesis and has written [3, p. 566]: "We shall need to employ the continuum hypothesis only in the last step of our argument."

The mentioned last step of the proof of Darst is a part of Corollary 3. Mauldin has proved that the continuum hypothesis can be replaced there by MA [9, p. 60–61].

REMARK 1. The implication that (i′) imples (ii′) can also be proved easily directly.

REMARK 2 (ADDED MARCH 15, 1980). In 1979 G. V. Cox (*A universal null graph whose domain has positive measure*, Colloq. Math. (to appear)), assuming the continuum hypothesis, answered both our questions. Namely he proved, in ZFC + CH, the existence of two uncountable subsets X_1, X_2 of R such that each uncountable subset of X_1 and each uncountable subset of X_2 have positive outer Lebesgue measure, (ii) holds for X_1 and (ii) does not hold for X_2.

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