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## SHORTER NOTES

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# A REGULAR SPACE WHICH IS NOT COMPLETELY REGULAR

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ABSTRACT. We give a simple example of a regular space which is not completely regular.

Every completely regular space is a regular space. The converse is false, however the construction of known counterexamples are quite complicated (compare **[4]** and [**2**, Example 2.4.21]). The aim of this note is to provide an elementary construction of a regular noncompletely regular space.

The underlying set of our space $X$ is the closed upper half-plane $y \geqslant 0$ plus an additional point $a$. All points $(x, y)$ with $y > 0$ are assumed to be isolated. The basic neighborhoods of $(x, 0)$ contain $(x, 0)$ and all but finitely many points from the union of two segments $I_x = \{(x, y)\colon 0 \leqslant y < 2\}$ and $I'_x = \{(x + y, y)\colon 0 \leqslant y < 2\}$. And the basic neighborhoods of the point $a$ have the form $U_n(a) = \{a\} \cup \{(x, y)\colon x > n\}$ where $n = 1, 2, \ldots$.

It is easy to check that $X$ is a regular space–neighborhoods of points from the half-plane are closed and open sets and $\mathrm{cl}_X\, U_{n+2}(a) \subset U_n(a)$ for every natural number $n$.

We shall prove that $X$ is not completely regular. The set $A = \{(x, 0)\colon x \leqslant 1\}$ is closed in $X$. Let $f$ be an arbitrary continuous real-valued function on $X$ such that $f(A) = \{0\}$. We show that $f(a) = 0$. Since the set $f^{-1}(0)$ is closed in $X$ it suffices to prove that for every natural number $n$ the set $K_n = f^{-1}(0) \cap \{(x, 0)\colon n - 1 \leqslant x \leqslant n\}$ is infinite. We proceed by induction. Obviously, the set $K_1 = \{(x, 0)\colon 0 \leqslant x \leqslant 1\}$ is infinite. Assume now that there is a countably infinite subset $C$ of $K_n$. For each $(c, 0) \in C$ the set $I'_c - f^{-1}(0)$ is an $F_\sigma$ set not containing $(c, 0)$ and therefore it is countable. Hence the projection $P$ of the union $\bigcup \{I'_c - f^{-1}(0)\colon (c, 0) \in C\}$ onto the line $y = 0$ is countable. Let $F = \{(x, 0)\colon n \leqslant x \leqslant n + 1\} - P$. Clearly $F$ is infinite. For every $(x, 0) \in F$ the segment $I_x$ meets each of the sets $I'_c \cap f^{-1}(0)$ with $(c, 0) \in C$, so that by the closedness of $f^{-1}(0)$ we have $(x, 0) \in f^{-1}(0)$. It follows that $F \subset f^{-1}(0)$. Thus the set $K_{n+1}$ is infinite and the proof is concluded.

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REMARK. If we adjoin to the space $X$ a new point $b$ with the basic neighborhoods of the form $U_n(b) = \{b\} \cup \{(x, y): x < -n \text{ and } x + n + 2 < y\}$, $n = 1, 2, \ldots$, then the obtained space $Y$ is regular and $f(a) = f(b)$ for every continuous real-valued function $f$ on $Y$. Using the space $Y$ and the ideas from **[1]** or **[3]** one can obtain a relatively simple example of a regular space on which all continuous real-valued functions are constant.

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