GENERALIZATIONS OF CERTAIN FUNDAMENTAL RESULTS ON FINITE GROUPS

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In this short note, we synthesize various fundamental results of finite groups and their proofs to obtain a generalization of J. G. Thompson's fundamental $A \times B$ Lemma (cf. [5, Theorem 5.3.4]), a generalization of a fundamental result of H. Bender (cf. [2, Satz] and [4, Theorem 3.1]) and a generalization of an important result of N. Blackburn (cf. [3, Theorem]).

Our notation is standard and tends to follow the notation of [5]. In particular, $p$ denotes a prime integer and all groups that we consider are finite.

Our first four results concern the following basic situation.

$V$ is a $p$-group acted on by the group $G$ where $G = KP$ with $K = O_p(G)$ and $P \in \text{Syl}_p(G)$.

We begin by observing that the proof of [2, Satz] yields a generalization of [5, Theorem 5.2.4].

**Theorem 1.** Suppose that $V$ is abelian. Let $k \in K$ be such that $\Omega_1(C_V(P)) < Cy(k)$. Then $[V, k] = 1$.

**Proof.** By [5, Theorem 5.2.4] and the observation that $\Omega_1(C_V(P)) = C_{\Omega_1}(V)$, we may assume that $V = \Omega_1(V)$. Moreover, by replacing $G$ by $G = G/C_V(V)$, we may assume that $C_V(V) = 1$. Also $k = C_V(C_V(P))$ is $P$-invariant and hence we may assume that $K = K_1$ and $C_V(P) < C_V(G)$. But $V = C_V(K) \times [V, K]$ and $P$ acts on $[V, K]$ by [5, Theorem 5.2.3]. Suppose that $[V, K] = 1$. Then $1 = C_V(P) \cap [V, K] < C_V(K)$ which is a contradiction. Thus $[V, K] = 1$ and we are done.

The next result also applies the proof of [2, Satz] to generalize the fundamental results [5, Theorems 5.3.4 and 5.3.10] and [4, Theorem 2.4] when $p \neq 2$.

**Theorem 2.** Suppose that $p \neq 2$. Let $k \in K$ be such that $\Omega_1(C_V(P)) < C_V(k)$. Then $[V, k] = 1$.

**Proof.** As above, we may assume that $C_V(V) = 1$ and $\Omega_1(C_V(P)) < C_V(G)$. Then [5, Theorem 5.3.13] implies that $V$ contains a characteristic subgroup $D$ of class at most 2 and exponent $p$ such that $K$ acts faithfully on $D$. Then $C_D(P) < \Omega_1(C_V(P)) < C_V(G)$ and hence we may assume that $V = D$. As in [2, (B)], we now use an observation of R. Baer (cf. [1, Theorem B.1]) to conclude that we may assume that $V$ is abelian. But then $C_V(P) < C_V(G)$ and an application of Theorem 1 completes the proof.
Note that Theorem 2 is false for $p = 2$. We give two examples.

(1) Let $V$ be a direct product of $n > 1$ quaternion groups of order 8 and let $G = KP$ where $K$ is an elementary abelian group of order $3^r$, $P$ consists of an involution inverting every element of $K$ and where $C_V(P) = C_V(K) = \Omega_1(V)$.

(2) Let $V$ be a 2-group of type $U_3(4)$ and let $G = KP < \text{Aut}(V)$ with $G$ a Frobenius group of order 20 such that $C_V(P) < C_V(K) = Z(V) = \Omega_1(V)$, (cf. [6, VI, Lemma 2.5]).

However, utilizing the proof of [4, Theorem 3.1], we can demonstrate

**Theorem 3.** Suppose that $p = 2$. Let $k \in K$ be such that $[V, P, \Omega_2(C_V(P))] < C_V(k)$. Then $[V, k] = 1$.

**Proof.** As above, we may assume that $C_k(V) = 1$, $\langle [V, P], \Omega_2(C_V(P)) \rangle < C_V(K)$ and $\Omega_2(C_V(P)) < C_V(G)$. Also, since $G$ acts on $C_V(K)$ and on $N_V(C_V(K))$, we may assume that $C_V(K) \leq V$. As $V = [V, K]C_V(K)$ by [5, Theorem 5.3.5], $[V, K] = [V, K, K]$ by [5, Theorem 5.3.6] and $(V, K)$ is $G$-invariant by [5, Theorem 2.2.1(iii)], we may assume that $V = [V, K]$. Theorem 1 implies that $K$ acts trivially on every characteristic abelian subgroup of $V$. Hence [5, Exercise 5.4] implies that $V$ is a nonabelian special 2-group. Thus $\exp(V) = 4$ and $C_V(P) < C_V(G)$. On the other hand, $K = [K, P]C_K(P)$ and $P \times C_K(P)$ acts on $V$. Since $C_K(V) = 1$, [5, Theorem 5.3.4] implies that $C_K(P) = 1$ and $[K, P] = [V, P] < C_V(K)$. Hence $[P, K, V] = [K, V] < C_V(K) \leq V$ by [5, Theorem 2.2.3(ii)]. Thus $K$ stabilizes the chain $V > C_V(K) > 1$ and [5, Theorem 5.3.2] forces $K = 1$ to complete the proof of Theorem 3.

It is easy to see that Theorem 3 implies Thompson’s $A \times B$ Lemma [5, Theorem 5.3.4] when $p = 2$. Suppose in the above that $G = K \times P$, $p = 2$, and $C_V(P) < C_V(K)$ for some $k \in K$. Proceed by induction on $|V|$. Since $[V, P] < V$ and $G$ acts on $[V, P]$, we conclude that $[V, P] < C_V(k)$. Then Theorem 3 yields the desired conclusion $[V, k] = 1$.

**Corollary 3.1.** Suppose that $p = 2$ and $[V, P]$ is contained in a characteristic abelian subgroup of $V$. Let $k \in K$ be such that $\Omega_2(C_V(P)) < C_V(k)$. Then $[V, k] = 1$.

**Proof.** Let $1 = M' < M$ char $V$. Thus $G$ acts on $M$ and $\Omega_1(C_M(P)) < \Omega_2(C_V(P)) < C_V(k)$. Then Theorem 1 implies that $M < C_V(k)$ and Theorem 3 yields the desired conclusion.

The next result generalizes and presents an alternate proof of [3, Theorem].

**Theorem 4.** Let $G$ be a finite $p$-group, let $E$ be a subgroup of $G$ and let $\alpha \in \text{Aut}(G)$ be such that $E < C_G(\alpha)$. Suppose that $\Omega_2(C_G(E)) < C_G(\alpha)$ if $p \neq 2$ and $\Omega_2(C_G(E)) < C_G(\alpha)$ if $p = 2$. Then the order of $\alpha$ is a power of $p$.

**Proof.** Suppose that $\alpha$ is also a $p'$-element of $\text{Aut}(G)$. Since $\langle \alpha \rangle \times E$ acts on $G$ and $\langle \alpha \rangle$ acts faithfully on $C_G(E)$ by [5, Theorem 5.3.4], we conclude that $\alpha = 1$ by Theorem 2 and Corollary 3.1 to complete the proof.

For our final result, we apply Theorems 2 and 3 to obtain generalizations of the fundamental results [2, Satz] and [4, Theorem 3.1] of H. Bender.
Theorem 5. Let $H$ be a $p$-constrained group, let $Q \in \text{Syl}_p(H)$, let $R = Q \cap O_{p',p}(H)$ and let $A$ be a subgroup of $Q$. Also let $K$ be an $A$-invariant $p'$-subgroup of $G$ and observe that this implies that $[A \cap R, K] < O_p(H)$. In addition, assume the following two conditions:

(a) if $p + 2$, then $[\Omega_1(C_R(A)), K] < O_p(H)$; and

(b) if $p = 2$, then $\langle [\Omega_2(C_R(A)), K], [R, A, K] \rangle < O_p(H)$.

Then $K < O_p(H)$.

Proof. Clearly we may assume that $O_p(H) = 1$ and $R = O_p(H)$. Then $KA$ acts on $R$, $C_H(R) = Z(R)$ and Theorems 2 and 3 immediately yield the desired conclusion.

References


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