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ON THE DUAL OF A CERTAIN OPERATOR IDEAL

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ABSTRACT. For complex Banach spaces E and F and a real number $1 < p < \infty$ let $S^p(E, F)$ denote the operator ideal obtained by complex interpolation between the nuclear and the compact operators. If E and F are reflexive and one of them has the approximation property the dual of $S^p(E, F)$ is shown to be $S^{p'}(E', F')$, p' conjugate to p.

Let N denote the ideal of nuclear and K the ideal of compact linear operators. By complex interpolation between N and K there is associated to every real number $1 \leqslant p \leqslant \infty$ the operator ideal S^p in [4], i.e., $S^p(E, F) = [N(E, F), K(E, F)]_{1/p'}$ for any complex Banach spaces E and F (that this definition gives the same as that of [4, p. 101] follows from [1, 9.3]). It is shown there that for any complex separable Hilbert space H, $S^p(H, H)$ consists of those compact operators whose moduli have pth power summable eigenvalues. It is then a classical result of Schatten and von Neumann that the Banach space dual of $S^p(H, H)$ may be identified with $S^{p'}(H, H)$. Here we are showing

PROPOSITION. *Let E and F be complex Banach spaces and p a real number with $1 < p < \infty$, $1/p' + 1/p = 1$. If E and F are reflexive and one of them has the approximation property then the dual of $S^p(E, F)$ may be identified isometrically with $S^{p'}(E', F')$, the pairing given by ($S \in S^{p'}(E', F')$ and $T \in S^p(E, F)$): $\langle T, S\rangle =$ trace$(T' \circ S)$ when E has the approximation property; $\langle T, S \rangle =$ trace$(S \circ T')$ when F has it.*

In particular the space $S^p(E, F)$ is reflexive.

PROOF. We shall use the notation of [1] without further explanation. Let us assume that E has the approximation property.

First step. Since E is reflexive E' has it too by [3 Proposition 36.1]. By Satz 3 and Satz 7 in [4], for every $S \in S^{p'}(E', F')$ and $T \in S^p(E, F)$ the product $T' \circ S$ is nuclear, and $\|T' \circ S\|_N \leqslant \|T'\|_{S^p}\|S\|_{S^{p'}} \leqslant \|T\|_{S^p}\|S\|_{S^{p'}}$. Since E' has the approximation property, trace$(T' \circ S)$ is well defined and $\leqslant \|T' \circ S\|_N \leqslant \|T\|_{S^p}\|S\|_{S^{p'}}$ such that β: $S^{p'}(E', F') \to S^p(E, F)'$, given by $\langle T, \beta S\rangle =$ trace$(T' \circ S)$, is a linear contraction.

Second step. β is an isometry. Since the linear mappings of finite rank are dense in $S^{p'}(E', F')$ it suffices to show that

$$\|S\|_{S^{p'}(E', F')} \leqslant \|\beta S\|_{S^p(E, F)'}$$

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for any such map. So let S: $E' \to F'$ be a linear map of finite rank. By the theorem of Hahn-Banach there exists a linear form L in $S^{p'}(E', F')'$ of norm 1 with $\|S\|_{S^{p'}} = \langle S, L\rangle = \operatorname{trace}(L' \circ S)$, when we identify L with the corresponding bounded linear map from E to F. By definition and the duality theorem 12.1 in **[1]** one has

$$S^{p'}(E', F')' = \left[N(E', F')', K(E', F')'\right]^{1/p}$$

since $N(E', F')$ is dense in $K(E', F')$ because of the approximation property of $E'' = E$. Identifying $N(E', F')' = (E'' \hat{\otimes} F')' = (E \hat{\otimes} F')' = H(E, F)$, the space of bounded linear mappings from E to F, and $K(E', F')' = (E'' \check{\otimes} F')' = (E \check{\otimes} F')' = I(E, F)$, the space of integral mappings from E to F, which coincides with $N(E, F)$, since F is reflexive [3, Théorème 10.1], we obtain

$$S^{p'}(E', F')' = \left[N(E, F), H(E, F)\right]^{t} \quad \text{with } t = 1 - 1/p = 1/p'.$$

So for every $\varepsilon > 0$ we can find a function h in $\bar{\mathcal{F}}(N(E, F), H(E, F))$ with $\|h\|_{\bar{\mathcal{F}}} < 1 + \varepsilon$ and whose derivative $h'(t)$ at the point t equals L.

For this function h we construct in the usual manner (cf. **[2]**) a sequence of functions h_n in $\mathcal{F}(N(E, F), H(E, F))$ with $\lim_{n\to\infty} h_n(t) = h'(t)$; for example,

$$h_n(z) = \exp(z^2/n)\left[h(z + i/n) - h(z)\right]n/i \quad \text{for } 0 < \operatorname{Re} z < 1, n > 1.$$

$\|h_n\|_{\mathcal{F}} \leqslant e^{1/n}\|h\|_{\bar{\mathcal{F}}}$ for every n. By 9.3 in **[1]** (second line from below) we have $\mathcal{F}(N(E, F), H(E, F)) = \mathcal{F}(N(E, F), K(E, F))$ isometrically so that $h_n(t) \in [N(E, F), K(E, F)]_t = S^p(E, F)$ and $\|h_n(t)\|_{S^p} \leqslant \|h_n\|_{\mathcal{F}} \leqslant e^{1/n}(1 + \varepsilon)$ for every n. All together one has

$$\begin{aligned}
\|S\|_{S^{p'}(E', F')} &= \operatorname{trace}(L' \circ S) = \operatorname{trace}\left(\left[h'(t)\right]' \circ S\right) \\
&= \lim \operatorname{trace}\left(\left[(h(t + i/n) - h(t))n/i\right]' \circ S\right) \\
&= \lim \operatorname{trace}\left(\exp(-t^2/n)\left[h_n(t)\right]' \circ S\right) \\
&= \lim \exp(-t^2/n)\langle h_n(t), \beta S\rangle \\
&\leqslant \lim\sup \exp(-t^2/n)\|h_n(t)\|_{S^p}\|\beta S\|_{S^p(E, F)'} \\
&\leqslant \lim\sup \exp((1 - t^2)/n)(1 + \varepsilon)\|\beta S\|_{S^p(E, F)'} \\
&\leqslant (1 + \varepsilon)\|\beta S\|_{S^p(E, F)'},
\end{aligned}$$

i.e.,

$$\|S\|_{S^{p'}(E', F')} \leqslant \|\beta S\|_{S^p(E,F)'}.$$

Since this inequality obtains for every $S \in S^{p'}(E', F')$, the isometry of β follows in conjunction with the contractivity of β.

Third step. The image of β is dense in $S^p(E, F)'$. First of all note that $S^p(E, F)'$ is equal to $[N(E, F)', K(E, F)']^{1/p'} = [N(E', F'), H(E', F')]^{1/p'}$ by the duality theorem (cf. Second step). Now by the last formula in 9.3 of **[1]** this last space is equal to $[N(E', F'), K(E', F')]^{1/p'}$ which by definition is contained in $N(E', F') + K(E', F') = K(E', F')$. Since $K(E', F')$ is the closure of the operators of finite rank and these are contained in $S^{p'}(E', F')$, β has dense image.

So β: $S^{p'}(E', F') \to S^p(E, F')$ is an isometric isomorphism. Since the proof works as well when F (hence F') has the approximation property the proposition is established.

Let now G denote a compact group with normalized Haar measure and $L^q(G)$ its complex Lebesgue spaces. Let $S_G^p(L^q(G), L^q(G))$ denote the set of those operators in $S^p(L^q(G), L^q(G))$ which are (left-or-right-) translation invariant under G. Since it is a closed subspace of $S^p(L^q(G), L^q(G))$ one arrives at

COROLLARY. *Let G be a compact group and $1 < p, q < \infty$. Then the two-sided q-Segal algebras $S_G^p(L^q(G), L^q(G))$ are reflexive Banach spaces.*

The assertion of the corollary remains true for $q = 1$, since then

$$S_G^p(L^1(G), L^1(G)) = L^{p'}(G),$$

which was the starting point for this note (cf. [5]).

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