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A PROOF OF THE BOUNDARY THEOREM

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ABSTRACT. This note contains a simple proof of the following theorem of Arveson: If $\mathcal{Q}$ is an irreducible subspace of $\mathcal{B}(H)$, then the identity map $\phi_0(A) = A$ on $\mathcal{Q}$ has a unique completely positive extension to $\mathcal{B}(H)$ if and only if the quotient map q by the compact operators is not completely isometric on $\mathcal{S} = [\mathcal{Q} + \mathcal{Q}^*]$.

Given a linear map ϕ: $\mathcal{Q} \to \mathcal{B}$ of one C^* algebra into another, we can form the maps $\phi \otimes \mathrm{id}_n$ of $n \times n$ matrices with coefficients in $\mathcal{Q}$ to $n \times n$ matrices with coefficients in $\mathcal{B}$ by taking (A_{ij}) to $(\phi(A_{ij}))$. The map ϕ is said to be completely positive if $\phi \otimes \mathrm{id}_n$ is positive for all n. These maps have proved to be of importance in the study of extensions of C^* algebras (e.g., [3], [4]), and in the study of nonselfadjoint subalgebras of C^* algebras [1], [2]. The difference between positive and completely positive maps has provided insight into the difference between positivity and sums of squares and Hilbert's seventeenth problem [5].

Stinespring [7] showed that complete positivity is intimately connected with the algebraic structure of the C^* algebra. He showed that if ϕ: $\mathcal{Q} \to \mathcal{B}(\mathcal{H})$ is a unital ($\phi(I) = I$), completely positive map of a C^* algebra $\mathcal{Q}$ into the bounded operators on a Hilbert space $\mathcal{H}$, then ϕ has the form $\phi(A) = V^*\pi(A)V$, where π is a * representation of $\mathcal{Q}$ on another Hilbert space $\mathcal{K}$ and V: $\mathcal{H} \to \mathcal{K}$ is an isometric embedding of $\mathcal{H}$ into $\mathcal{K}$. In general, positive maps are not this nice, but in commutative algebras every positive map is completely positive.

Arveson [1] recognized that $\mathcal{B}(\mathcal{H})$ is injective for completely positive maps. He proved that if ϕ is a completely positive map from a selfadjoint subspace (containing the identity) of a C^* algebra $\mathcal{Q}$ into $\mathcal{B}(\mathcal{H})$, then ϕ has a completely positive extension ϕ_1: $\mathcal{Q} \to \mathcal{B}(\mathcal{H})$. In his study [1], [2] of nonselfadjoint subalgebras of C^* algebras, he showed that completely positive maps on these subalgebras which have a unique completely positive extension of the whole C^* algebra play an important role. In the important special case of an irreducible subalgebra $\mathcal{Q}$ of $\mathcal{B}(\mathcal{H})$, it was shown that "sufficiently many" of these maps exist provided the identity map restricted to $\mathcal{Q}$ has a unique completely positive extension.

Let $\mathcal{Q}$ be an irreducible linear subspace of $\mathcal{B}(\mathcal{H})$, and let $\mathcal{S}$ be the closed linear span of $\mathcal{Q} \cup \mathcal{Q}^*$. A map ϕ: $\mathcal{Q} \to \mathcal{B}(\mathcal{H})$ is completely contractive if $\|\phi \otimes \mathrm{id}_n\| < 1$ for all n. Such a ϕ has a unique completely positive extension to $\mathcal{S}$, namely set $\phi(A^*) = \phi(A)^*$ and extend by linearity. Corresponding, every completely positive map with $\phi(I) = I$ is completely contractive. We say that ϕ is completely isometric

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if $\phi \otimes \mathrm{id}_n$ is isometric for all n. Let q denote the quotient map of $\mathfrak{B}(\mathfrak{H})$ onto the Calkin algebra $\mathfrak{B}(\mathfrak{H})/\mathcal{C}(\mathfrak{H})$ where $\mathcal{C}(\mathfrak{H})$ is the ideal of compact operators.

We can now state Arveson's "Boundary Theorem" **[2]** which gives necessary and sufficient conditions for the identity map on $\mathfrak{A}$ to have a unique completely positive extension to $\mathfrak{B}(\mathfrak{H})$ (namely the identity map). The purpose of this note is to provide a simpler proof of this thoerem.

THEOREM *The identity map* $\phi_0(A) = A$ *restricted to* $\mathfrak{A}$ *has a unique completely positive extension to* $\mathfrak{B}(\mathfrak{H})$ *if and only if* q *is not completely isometric on* $\mathcal{S} = [\mathfrak{A} + \mathfrak{A}^*]$.

PROOF. One direction is straightforward. If q is completely isometric on $\mathcal{S}$, then the map ψ_0: $q(A) = A$ is a completely positive map of $q(\mathcal{S})$ into $\mathfrak{B}(\mathfrak{H})$. By Arveson's extension theorem, there is a completely positive map ψ from the Calkin algebra into $\mathfrak{B}(\mathfrak{H})$ which extends ψ_0. Then $\phi = \psi \cdot q$ extends ϕ_0 and annihilates the compact operators; so it is not the identity map.

For the converse, let ϕ be any completely positive extension of ϕ_0. Since q is not completely isometric on $\mathcal{S}$, there is an integer n so that $q \otimes \mathrm{id}_n$ is not isometric on $\mathcal{S} \otimes \mathfrak{M}_n$. ($\mathfrak{M}_n$ denotes the $n \times n$ matrices over $\mathbb{C}$.) The map $\phi_0 \otimes \mathrm{id}_n$ has a completely positive extension $\phi \otimes \mathrm{id}_n$ to $\mathfrak{B}(\mathfrak{H}) \otimes \mathfrak{M}_n$ which is the identity map if and only if ϕ is the identity. So without loss of generality, we can suppose that q is not isometric on $\mathcal{S}$.

By Stinespring's theorem **[7]**, there is a representation π of $\mathfrak{B}(\mathfrak{H})$ on a Hilbert space $\mathfrak{K}$ and an isometry V: $\mathfrak{H} \to \mathfrak{K}$ such that $\phi(X) = V^*\pi(X)V$ for all X in $\mathfrak{B}(\mathfrak{H})$. $\mathcal{C}(\mathfrak{H})$ is a two-sided ideal in $\mathfrak{B}(\mathfrak{H})$ and its only irreducible representation is the identity representation. So π can be decomposed as $\pi = \pi_a \oplus \pi_s$ on $\mathfrak{K} = \mathfrak{K}_a \oplus \mathfrak{K}_s$ so that π_a is a multiple of the identity representation, and π_s annihilates the compact operators **[6**, §4.7.22]. We identify $\mathfrak{K}_a$ with a direct sum $\Sigma\mathfrak{H}$ of copies of $\mathfrak{H}$ via $\pi_a \cong n \cdot \mathrm{id}$, where n is some cardinal number. Also, we can factor $\pi_s = \dot{\pi}_s \circ q$.

Choose a T in $\mathcal{S}$ so that $\|T\| > \|q(T)\|$. Then there is a unit vector ξ such that $\|T\xi\| = \|T\|$. Furthermore, $\mathcal{E} = \{\xi\colon \|T\xi\| = \|T\| \cdot \|\xi\|\}$ is a finite dimensional subspace. To see this, write $T = U|T|$ in its polar decomposition. Then $\|\,|T|\,\| = \|T\| > \|q(T)\| = \|q(|T|)\|$. So the restriction of $|T|$ to the spectral subspace $E[\|q(T)\| + \varepsilon, \|T\|]$ is compact and nonzero. So the subspace $E[\|T\|]$ is nonempty and finite dimensional, and is precisely $\mathcal{E}$.

If $\xi \in \mathcal{E}$, then $V\xi \in \mathfrak{K}_a$. For if $V\xi = \nu_a \oplus \nu_s$,

$$\begin{aligned}\|T\xi\|^2 = \|\phi(T)\xi\|^2 &= \|V^*(\pi_a(T)\nu_a \oplus \dot{\pi}_s \circ q(T)\nu_s)\|^2 \\ &\leqslant \|T\|^2\|\nu_a\|^2 + \|q(T)\|^2\|\nu_s\|^2 \leqslant \|T\|^2\|\xi\|^2.\end{aligned}$$

The extreme terms are equal, so it follows that $\nu_s = 0$ and $\|\pi_a(T)\nu_a\| = \|T\|\|\xi\|$. Thus, $V\mathcal{E} \subseteq \bigoplus \Sigma\mathcal{E}$.

Let $\mathfrak{N}$ be a minimal nonzero subspace of $\mathcal{E}$ satisfying $V\mathfrak{N} \subseteq \bigoplus \Sigma\mathfrak{N}$. Let $\Gamma = \{X \in \mathfrak{B}(\mathfrak{H})\colon VX\nu = \pi(X)V\nu$ for all ν in $\mathfrak{N}\}$. Then Γ is a closed linear space containing the identity I. We will show that if X belongs to Γ and S belongs to $\mathcal{S}$, then SX belongs to Γ.

Let X and S be fixed, and set $\mathfrak{N}_0 = \{\nu \in \mathfrak{N}: \|SX\nu\| = \|SX|_{\mathfrak{N}}\| \cdot \|\nu\|\}$. If ν belongs to $\mathfrak{N}_0$, then

$$\begin{aligned} \|SX\nu\| &= \|\phi(S)X\nu\| = \|V^*\pi(S)VX\nu\| = \|V^*\pi(SX)V\nu\| \\ &\leq \|\pi(SX)|_{\oplus\Sigma\mathfrak{N}}\| \cdot \|\nu\| = \|SX|_{\mathfrak{N}}\| \cdot \|\nu\| = \|SX\nu\|. \end{aligned}$$

Hence $V\nu$ belongs to $\oplus\Sigma\mathfrak{N}_0$ and $V\mathfrak{N}_0 \subseteq \oplus\Sigma\mathfrak{N}_0$. By the minimality of $\mathfrak{N}$, we must have $\mathfrak{N} = \mathfrak{N}_0$. It also follows that $\|\pi(SX)V\nu\| = \|V^*\pi(SX)V\nu\| = \|VV^*\pi(SX)V\nu\|$. So

$$\pi(SX)V\nu = VV^*\pi(S)\pi(X)V\nu = VV^*\pi(S)VX\nu = V\phi(S)X\nu = VSX\nu.$$

This holds for all ν in $\mathfrak{N}_0 = \mathfrak{N}$, so SX belongs to Γ.

Since $\mathbb{S}$ is selfadjoint, Γ must contain $C^*(\mathbb{S})$. As noted earlier, the orthogonal projection onto $\mathcal{E}$ belongs to $C^*(\mathbb{S})$, so $C^*(\mathbb{S})$ contains a nonzero compact operator. Since $\mathcal{Q}$ is irreducible, $C^*(\mathbb{S})$ must contain all compact operators. If X and S are operators in $C^*(\mathbb{S})$,

$$\begin{aligned} XS\nu &= V^*VXS\nu = V^*\pi(XS)V\nu = V^*\pi(X)\pi(S)V\nu \\ &= V^*\pi(X)VS\nu = \phi(X)S\nu. \end{aligned}$$

But $C^*(\mathbb{S})$ is transitive, thus $\phi(X) = X$ for all X in $C^*(\mathbb{S})$.

Finally, since ϕ is the identity on the compact operators, $V\mathcal{H}$ must be contained in $\mathcal{H}_a$. Consequently, $\pi = \pi_a$ is ultra-weakly continuous. Hence ϕ is the identity on all of $\mathcal{B}(\mathcal{H})$.

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