A PROOF OF THE BOUNDARY THEOREM

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Abstract. This note contains a simple proof of the following theorem of Arveson: 
If $\mathcal{A}$ is an irreducible subspace of $\mathcal{B}(H)$, then the identity map $\phi(A) = A$ on $\mathcal{A}$ has a unique completely positive extension to $\mathcal{B}(H)$ if and only if the quotient map $q$ by the compact operators is not completely isometric on $S = [\mathcal{A} + \mathcal{A}^*]$.

Given a linear map $\phi: \mathcal{A} \to \mathcal{B}$ of one C*-algebra into another, we can form the maps $\phi \otimes \text{id}_n$ of $n \times n$ matrices with coefficients in $\mathcal{A}$ to $n \times n$ matrices with coefficients in $\mathcal{B}$ by taking $(A_0)$ to $(\phi(A_0))$. The map $\phi$ is said to be completely positive if $\phi \otimes \text{id}_n$ is positive for all $n$. These maps have proved to be of importance in the study of extensions of C*-algebras (e.g., [3], [4]), and in the study of nonselfadjoint subalgebras of C*-algebras [1], [2]. The difference between positive and completely positive maps has provided insight into the difference between positivity and sums of squares and Hilbert’s seventeenth problem [5].

Stinespring [7] showed that complete positivity is intimately connected with the algebraic structure of the C*-algebra. He showed that if $\phi: \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a unital $\phi(I) = I$, completely positive map of a C*-algebra $\mathcal{A}$ into the bounded operators on a Hilbert space $\mathcal{K}$, then $\phi$ has the form $\phi(A) = V^*\pi(A)V$, where $\pi$ is a *-representation of $\mathcal{A}$ on another Hilbert space $\mathcal{K}$ and $V: \mathcal{K} \to \mathcal{K}$ is an isometric embedding of $\mathcal{K}$ into $\mathcal{K}$. In general, positive maps are not this nice, but in commutative algebras every positive map is completely positive.

Arveson [1] recognized that $\mathcal{B}(\mathcal{K})$ is injective for completely positive maps. He proved that if $\phi$ is a completely positive map from a selfadjoint subspace (containing the identity) of a C*-algebra $\mathcal{A}$ into $\mathcal{B}(\mathcal{K})$, then $\phi$ has a completely positive extension $\phi_I: \mathcal{A} \to \mathcal{B}(\mathcal{K})$. In his study [1], [2] of nonselfadjoint subalgebras of C*-algebras, he showed that completely positive maps on these subalgebras which have a unique completely positive extension of the whole C*-algebra play an important role. In the important special case of an irreducible subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{K})$, it was shown that “sufficiently many” of these maps exist provided the identity map restricted to $\mathcal{A}$ has a unique completely positive extension.

Let $\mathcal{A}$ be an irreducible linear subspace of $\mathcal{B}(\mathcal{K})$, and let $S$ be the closed linear span of $\mathcal{A} \cup \mathcal{A}^*$. A map $\phi: \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is completely contractive if $\|\phi \otimes \text{id}_n\| < 1$ for all $n$. Such a $\phi$ has a unique completely positive extension to $S$, namely set $\phi(A^*) = \phi(A)^*$ and extend by linearity. Corresponding, every completely positive map with $\phi(I) = I$ is completely contractive. We say that $\phi$ is completely isometric.
if \( \phi \otimes \text{id}_n \) is isometric for all \( n \). Let \( q \) denote the quotient map of \( \mathcal{B}(\mathcal{K}) \) onto the Calkin algebra \( \mathcal{B}(\mathcal{K})/\mathcal{C}(\mathcal{K}) \) where \( \mathcal{C}(\mathcal{K}) \) is the ideal of compact operators.

We can now state Arveson’s “Boundary Theorem” [2] which gives necessary and sufficient conditions for the identity map on \( \mathcal{E} \) to have a unique completely positive extension to \( \mathcal{B}(\mathcal{K}) \) (namely the identity map). The purpose of this note is to provide a simpler proof of this theorem.

**Theorem** The identity map \( \phi_0(A) = A \) restricted to \( \mathcal{E} \) has a unique completely positive extension to \( \mathcal{B}(\mathcal{K}) \) if and only if \( q \) is not completely isometric on \( \mathbb{S} = [\mathcal{E} + \mathcal{E}^*] \).

**Proof.** One direction is straightforward. If \( q \) is completely isometric on \( \mathbb{S} \), then the map \( \psi_0: q(A) = A \) is a completely positive map of \( q(\mathbb{S}) \) into \( \mathcal{B}(\mathcal{K}) \). By Arveson’s extension theorem, there is a completely positive map \( \psi \) from the Calkin algebra into \( \mathcal{B}(\mathcal{K}) \) which extends \( \psi_0 \). Then \( \phi = \psi \cdot q \) extends \( \phi_0 \) and annihilates the compact operators; so it is not the identity map.

For the converse, let \( \phi \) be any completely positive extension of \( \phi_0 \). Since \( q \) is not completely isometric on \( \mathbb{S} \), there is an integer \( n \) so that \( q \otimes \text{id}_n \) is not isometric on \( \mathbb{S} \otimes \mathcal{M}_n \) (\( \mathcal{M}_n \) denotes the \( n \times n \) matrices over \( \mathbb{C} \)). The map \( \phi_0 \otimes \text{id}_n \) has a completely positive extension \( \phi \otimes \text{id}_n \) to \( \mathcal{B}(\mathcal{K}) \otimes \mathcal{M}_n \) which is the identity map if and only if \( \phi \) is the identity. So without loss of generality, we can suppose that \( q \) is not isometric on \( \mathbb{S} \).

By Stinespring’s theorem [7], there is a representation \( \pi \) of \( \mathcal{B}(\mathcal{K}) \) on a Hilbert space \( \mathcal{K} \) and an isometry \( V: \mathcal{K} \rightarrow \mathcal{K} \) such that \( \phi(X) = V^* \pi(X) V \) for all \( X \) in \( \mathcal{B}(\mathcal{K}) \). \( \mathcal{C}(\mathcal{K}) \) is a two-sided ideal in \( \mathcal{B}(\mathcal{K}) \) and its only irreducible representation is the identity representation. So \( \pi \) can be decomposed as \( \pi = \pi_a \oplus \pi_s \) on \( \mathcal{K} = \mathcal{K}_a \oplus \mathcal{K}_s \) so that \( \pi_a \) is a multiple of the identity representation, and \( \pi_s \) annihilates the compact operators [6, §4.7.22]. We identify \( \mathcal{K}_a \) with a direct sum \( \Sigma \mathcal{K} \) of copies of \( \mathcal{K} \) via \( \pi_a = n \cdot \text{id} \), where \( n \) is some cardinal number. Also, we can factor \( \pi_s = \pi_s \circ q \).

Choose a \( T \) in \( \mathbb{S} \) so that \( ||T|| > ||q(T)|| \). Then there is a unit vector \( \xi \) such that \( ||T\xi|| = ||T|| \). Furthermore, \( \mathcal{E} = \{ \xi: ||T\xi|| = ||T|| \cdot ||\xi|| \} \) is a finite dimensional subspace. To see this, write \( T = U|T| \) in its polar decomposition. Then \( |||T||| = ||q(T)|| = \pi(|T|) ||q(T)|| \). So the restriction of \( |T| \) to the spectral subspace \( E(||q(T)|| + \varepsilon, ||T||) \) is compact and nonzero. So the subspace \( E(||T||) \) is nonempty and finite dimensional, and is precisely \( \mathcal{E} \).

If \( \xi \in \mathcal{E} \), then \( V\xi \in \mathcal{K}_a \). For if \( V\xi = v_a \oplus v_s \),

\[
||T\xi||^2 = ||p(T)\xi||^2 = ||V^*(\pi_a(T)v_a \oplus \pi_s \circ q(T)v_s)||^2 \\
< ||T||^2||v_a||^2 + ||q(T)||^2||v_s||^2 < ||T||^2||\xi||^2.
\]

The extreme terms are equal, so it follows that \( v_s = 0 \) and \( ||\pi_a(T)v_a|| = ||T||||\xi||. \) Thus, \( V\mathcal{E} \subseteq \bigoplus \Sigma \mathcal{E} \).

Let \( \mathcal{R} \) be a minimal nonzero subspace of \( \mathcal{E} \) satisfying \( V\mathcal{R} \subseteq \bigoplus \Sigma \mathcal{R} \). Let \( \Gamma = \{ X \in \mathcal{B}(\mathcal{K}): VXv = \pi(X) Vv \text{ for all } v \in \mathcal{R} \} \). Then \( \Gamma \) is a closed linear space containing the identity \( I \). We will show that if \( X \) belongs to \( \Gamma \) and \( \mathcal{S} \) belongs to \( \mathbb{S} \), then \( SX \) belongs to \( \Gamma \).
Let $X$ and $S$ be fixed, and set $\mathcal{N}_0 = \{ \nu \in \mathcal{H} : \|SX\nu\| = \|SX\| \cdot \|\nu\| \}$. If $\nu$ belongs to $\mathcal{N}_0$, then
\[
\|SX\nu\| = \|\phi(S)X\nu\| = \|V^*\pi(S) VX\nu\| = \|V^*\pi(SX) V\nu\| < \|\pi(SX) \| \cdot \|\nu\| = \|SX\| \cdot \|\nu\| = \|SX\nu\|.
\]
Hence $V\nu$ belongs to $\bigoplus \mathcal{N}_0$ and $V\mathcal{N}_0 \subseteq \bigoplus \mathcal{N}_0$. By the minimality of $\mathcal{N}$, we must have $\mathcal{N} = \mathcal{N}_0$. It also follows that $\|\pi(SX) V\nu\| = \|V^*\pi(SX) V\nu\| = \|V V^*\pi(SX) V\nu\|$. So
\[
\pi(SX) V\nu = VV^*\pi(S)\pi(X) V\nu = VV^*\pi(S) VX\nu = V\phi(S) X\nu = VXS\nu.
\]
This holds for all $\nu$ in $\mathcal{N}_0 = \mathcal{N}$, so $SX$ belongs to $\Gamma$.

Since $S$ is selfadjoint, $\Gamma$ must contain $C^*(S)$. As noted earlier, the orthogonal projection onto $S$ belongs to $C^*(S)$, so $C^*(S)$ contains a nonzero compact operator. Since $\mathcal{H}$ is irreducible, $C^*(S)$ must contain all compact operators. If $X$ and $S$ are operators in $C^*(S)$,
\[
XS\nu = V^*VXS\nu = V^*\pi(XS) V\nu = V^*\pi(X)\pi(S) V\nu = V^*\pi(X) VS\nu = \phi(X) S\nu.
\]
But $C^*(S)$ is transitive, thus $\phi(X) = X$ for all $X$ in $C^*(S)$.

Finally, since $\phi$ is the identity on the compact operators, $V\mathcal{K}$ must be contained in $\mathcal{K}_0$. Consequently, $\pi = \pi_a$ is ultra-weakly continuous. Hence $\phi$ is the identity on all of $B(\mathcal{H})$.

**REFERENCES**


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