PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 1, May 1981

AN ABSTRACT LINEAR VOLTERRA EQUATION WITH A NONCONVOLUTION KERNEL

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ABSTRACT. This paper is concerned with the existence and uniqueness of solutions to the equation $x(t) + \int_0^t a(t, \tau)Ax(\tau)\, d\tau = f(t)$ where A is an unbounded, positive, selfadjoint operator on a Hilbert space. A representation is given for the solution of this equation.

1. Introduction. In this paper we will be concerned with the existence and uniqueness of solutions to the equation

$$x(t) + \int_0^t a(t, \tau)Ax(\tau)\, d\tau = f(t), \qquad 0 < t < T, \tag{1.1}$$

where A is an unbounded, positive, selfadjoint operator on a Hilbert space H, $a(t, \tau)$ is a real-valued function and x, f: $[0, T] \to H$. Our goal is to extend the existence and uniqueness results of Clement and Nohel [1], Friedman and Shinbrot [2], and Kiffe and Stecher [7] for the convolution equation

$$x(t) + \int_0^t b(t - \tau)Ax(\tau)\, d\tau = f(t), \qquad 0 < t < T, \tag{1.2}$$

to the nonconvolution equation (1.1).

Our approach to solving (1.1) will follow that of [1], [7] and consists of first considering the properties of the solutions of the resolvent scalar equations

$$r_\lambda(t, \tau) + \lambda \int_\tau^t a(t, u)r_\lambda(u, \tau)\, du = a(t, \tau) \tag{1.3}$$

and

$$s_\lambda(t, \tau) + \lambda \int_\tau^t a(t, u)s_\lambda(u, \tau)\, du = 1. \tag{1.4}$$

If we define resolvent operators $R(t, \tau)$ and $S(t, \tau)$ by

$$R(t, \tau) = \int_0^\infty r_\lambda(t, \tau)\, dE(\lambda) \tag{1.5}$$

and

$$S(t, \tau) = \int_0^\infty s_\lambda(t, \tau)\, dE(\lambda), \tag{1.6}$$

where $\{E(\lambda) | \lambda \geq 0\}$ is the resolution of the identity determined by A, it will be shown that solutions of (1.1) can be written in the form

$$x(t) = f(t) - \int_0^t R(t, \tau)Af(\tau)\, d\tau, \tag{1.7}$$

Received by the editors March 27, 1980.
AMS (MOS) subject classifications (1970). Primary 45D05, 45N05; Secondary 47B25.

or

$$x(t) = S(t, 0)f(0) + \int_0^t S(t, \tau)f'(\tau)\, d\tau \tag{1.8}$$

under suitable assumptions on a, f, and f'. For other related results on linear Volterra equations in abstract spaces we refer the interested reader to [5], [6].

2. Statement and discussion of results. Throughout this paper H will denote a real Hilbert space with norm $|\cdot|$ and inner product $\langle \cdot, \cdot \rangle$ and

$$L^1[0, T; H] = \left\{ f\colon [0, T] \to H \,|\, f \text{ is strongly measurable and } \int_0^T |f(t)|\, dt < \infty \right\}.$$

A will always denote an unbounded, positive, linear, selfadjoint operator from H to H with dense domain $D(A)$ and $\{E(\lambda) | \lambda \geqslant 0\}$ will denote the resolution of the identity determined by A. For the standard results concerning the resolution of the identity and the spectral theorem for selfadjoint operators we refer the reader to [10]. We shall also set $H_\alpha = D(A^\alpha)$ for $0 \leqslant \alpha < \infty$ and if we define a norm on H by $\|x\|_\alpha = |x| + |A^\alpha x|$ then H_α becomes a Hilbert space itself.

Next we define precisely what we mean by a solution of (1.1). A function x: $[0, T] \to H$ is a strong solution of (1.1) if $x \in L^1[0, T; H]$, $x(t) \in D(A)$ a.e. on $[0, T]$, $Ax \in L^1[0, T; H]$ and $x(t)$ satisfies (1.1) on $[0, T]$. Later we will define a weak solution of (1.1).

Concerning the kernel $a(t, \tau)$ we shall assume

(i) $a(t, \tau)$ is continuous for $0 \leqslant \tau \leqslant t \leqslant T$ and is absolutely continuous in t for each fixed τ, $\tau \leqslant t \leqslant T$;

(ii) $0 < \varepsilon \leqslant a(t, \tau)$ for $0 \leqslant \tau \leqslant t \leqslant T$ for some constant ε and $(\partial/\partial t)a(t, \tau) \leqslant 0$ for $0 \leqslant \tau \leqslant t \leqslant T$;

(iii) $a(t, t) + \int_0^t (\partial/\partial t)a(t, \tau)\, d\tau \geqslant 0$ for $0 \leqslant t \leqslant T$;

(iv) for each $\lambda \geqslant 0$ the solution of (1.3) satisfies $r_\lambda(t, \tau) \geqslant 0$ for $0 \leqslant \tau \leqslant t \leqslant T$.

THEOREM 1. *Suppose $a(t, \tau)$ satisfies* (i)–(iv). *If $f = f_1 + f_2$ where $f_1 \in L^1[0, T; H_{1+\alpha}]$ and $f_2 \in W^{1,1}[0, T; H_\alpha]'$ for some α, $0 < \alpha \leqslant 1$, then* (1.1) *has a unique strong solution $x(t) = x_1(t) + x_2(t)$ where $x_1(t)$ is given by* (1.7) *with $f = f_1$ and $x_2(t)$ is given by* (1.8) *with $f = f_2$. Furthermore there is a constant $c = c(T) > 0$ depending only on a, α and T such that*

$$\|x\|_{L^1[0,T;H]} \leqslant c\left\{ \|f_1\|_{L^1[0,T;H_\alpha]} + \|f_2\|_{W^{1,1}[0,T;H]} \right\}. \tag{2.1}$$

In Theorem 1, $W^{1,1}$ is the usual Sobolev space and concerning the hypothesis on $a(t, \tau)$ we remark that (i) and (ii) imply that for each $\lambda \geqslant 0$ the solution of (1.4) satisfies $s_\lambda(t, \tau) \geqslant 0$ (see (3.2)–(3.4) below). A sufficient condition on $a(t, \tau)$ which insures (iv) is given in [4, Theorem 1] which under suitable differentiability conditions is equivalent to $(\partial/\partial\tau)(\partial/\partial t)\log a(t, \tau) \leqslant 0$. Condition (iii) is a technical assumption needed to handle nonconvolution kernels. These assumptions on $a(t, \tau)$ are the natural extension to nonconvolution kernels of the conditions imposed on $b(t)$ in (1.2) in [1], [7]. In [2] Laplace transform methods are used to study (1.2) and hence their methods have no direct extension to convolution equations. Also in [1], [2] (1.2) is studied in a more general setting than used here for (1.1).

Similar to [1] we define a weak solution of (1.1) as follows. A function x: $[0, T] \to H$ is a weak solution of (1.1) if there are sequences $\{x_n\}$ and $\{f_n\}$ where each $f_n \in L^1[0, T; H]$ and each x_n is a strong solution of (1.1) with $f = f_n$ such that $f_n \to f$ and $x_n \to x$ in $L^1[0, T; H]$. It is immediate from (2.1) that (1.1) has a unique weak solution if $f \in L^1[0, T; H_\alpha] + W^{1,1}[0, T; H]$ for some α, $0 < \alpha < 1$, given by (1.7) and (1.8). (Note that $L^1[0, T; H_{1+\alpha}]$ is dense in $L^1[0, T; H_\alpha]$ with respect to the norm in $L^1[0, T; H_\alpha]$; similarly $W^{1,1}[0, T; H_\alpha]$ is dense in $W^{1,1}[0, T; H]$.)

If f satisfies the hypotheses of Theorem 1 with $\alpha = 1$ (i.e., $f \in L^1[0, T; H_2] + W^{1,1}[0, T; H_1]$) then the hypotheses on $a(t, \tau)$ can be significantly weakened.

THEOREM 2. *Suppose $a(t, \tau)$ is positive and continuous for $0 \leq \tau \leq t \leq T$, that for each fixed τ, $a(t, \tau)$ is a nonincreasing function of t, and that $\int_0^t a(t, \tau)\, d\tau$ is absolutely continuous for $0 \leq t \leq T$. If $f \in L^1[0, T; H_2] + W^{1,1}[0, T; H_1]$ then* (1.1) *has a unique strong solution $x(t) = x_1(t) + x_2(t)$ where x_1 and x_2 are as given in Theorem* 1 *and $x(t)$ satisfies* (2.1).

Theorem 2 now implies that (1.1) has a unique weak solution given by (1.7) and (1.8) if $f \in L^1[0, T; H_1] + W^{1,1}[0, T; H]$ if $a(t, \tau)$ is continuous, positive and nonincreasing in t. The proof of Theorem 2 uses a remarkable inequality due to Levin [8].

3. Proofs. Let $r_\lambda(t, \tau)$ and $s_\lambda(t, \tau)$ be the unique solutions of (1.3) and (1.4) respectively. By Theorem 3.1 of [9] $r_\lambda(t, \tau)$ and $s_\lambda(t, \tau)$ are continuous in (t, τ) for each fixed λ and a direct application of Gronwall's inequality shows that $r_\lambda(t, \tau)$ and $s_\lambda(t, \tau)$ are also continuous in λ. Also a direct substitution establishes that

$$s_\lambda(t, \tau) = 1 - \lambda \int_\tau^t r_\lambda(t, u)\, du. \tag{3.1}$$

By hypothesis $r_\lambda(t, \tau) \geq 0$ for $\lambda \geq 0$, $0 \leq \tau \leq t \leq T$. Next we show that

$$s_\lambda(t, \tau) > 0 \quad \text{for } \lambda \geq 0, 0 \leq \tau \leq t \leq T. \tag{3.2}$$

Fix τ and λ. Suppose (3.2) is false. Since $s_\lambda(\tau, \tau) = 1$ there is a number $t_0 > \tau$ so that $s_\lambda(t_0, \tau) = 0$ but $s_\lambda(t, \tau) > 0$ for $\tau \leq t < t_0$. Hence we must have that $(\partial/\partial t)s_\lambda(t_0, \tau) \leq 0$. Now differentiate (1.4) with respect to t and evaluate at $t = t_0$ to obtain

$$\frac{\partial}{\partial t} s_\lambda(t_0, \tau) + \lambda a(t_0, t_0) s_\lambda(t_0, \tau) + \lambda \int_\tau^{t_0} \frac{\partial}{\partial t} a(t_0, u) s_\lambda(u, \tau)\, d\tau = 0. \tag{3.3}$$

By (ii) we obtain $(\partial/\partial t)s_\lambda(t_0, \tau) > 0$ unless $(\partial/\partial t)a(t_0, u) = 0$ a.e. for $\tau \leq u \leq t_0$. But then we obtain

$$(\partial/\partial t)s_\lambda(t, \tau) + \lambda a(t, t) s_\lambda(t, \tau) = 0 \quad \text{for } \tau \leq t \leq t_0. \tag{3.4}$$

Solving for $s_\lambda(t, \tau)$ in (3.4) we obtain $s_\lambda(t_0, \tau) = C \exp(-\int_\tau^{t_0} a(u, u)\, du) > 0$, again a contradiction. This establishes (3.2).

Next we wish to show that there are positive constants C_α for $0 < \alpha < 1$, independent of (t, τ), so that

$$\sup_{\lambda > 0} \lambda^\alpha r_\lambda(t, \tau) < C_\alpha a(t, \tau)(t - \tau)^{-\alpha} \tag{3.5}$$

and

$$\sup_{\lambda > 0} \lambda^\alpha s_\lambda(t, \tau) < C_\alpha (t - \tau)^{-\alpha}. \tag{3.6}$$

By Theorem 2.7 of [9] we can rewrite (1.3) as

$$r_\lambda(t, u) = a(t, u) - \lambda \int_u^t r_\lambda(t, o) a(o, u)\, do. \tag{3.7}$$

Integrating (3.7) in u from τ to t and interchanging the order of integration we obtain

$$\int_\tau^t r_\lambda(t, u)\, du = \int_\tau^t a(t, u)\, du - \lambda \int_\tau^t r_\lambda(t, o) \left[\int_\tau^o a(o, u)\, du \right] do. \tag{3.8}$$

By (iii) we have $\int_\tau^o a(o, u)\, du$ is an increasing function of o and since $r_\lambda(t, \tau) > 0$ (3.8) implies

$$\int_\tau^t r_\lambda(t, u)\, du > \int_\tau^t a(t, u)\, du - \lambda \left[\int_\tau^t a(t, u)\, du \right] \left[\int_\tau^t r_\lambda(t, o)\, do \right]. \tag{3.9}$$

Solving (3.9) we get

$$\int_\tau^t r_\lambda(t, u)\, du > \left[\int_\tau^t a(t, u)\, du \right] \left[1 + \lambda \int_\tau^t a(t, u)\, du \right]^{-1}. \tag{3.10}$$

On the other hand the second part of (ii) implies

$$\lambda \int_\tau^t r_\lambda(t, u)\, du < \frac{\lambda \int_\tau^t r_\lambda(t, u) a(u, \tau)\, du}{a(t, \tau)} = \frac{a(t, \tau) - r_\lambda(t, \tau)}{a(t, \tau)}$$

and hence we have

$$r_\lambda(t, \tau) < a(t, \tau) \left[1 - \lambda \int_\tau^t r_\lambda(t, u)\, du \right]. \tag{3.11}$$

Combining (3.10) and (3.11) we obtain

$$r_\lambda(t, \tau) < a(t, \tau) \left[1 + \lambda \int_\tau^t a(t, u)\, du \right]^{-1}. \tag{3.12}$$

(Note that (3.12) extends Theorem 1 of [3].) By (3.1) and (3.12) we obtain

$$s_\lambda(t, \tau) < \left[1 + \lambda \int_\tau^t a(t, u)\, du \right]^{-1}. \tag{3.13}$$

Now multiplying (3.12) and (3.13) by λ^α, maximizing in λ for $0 < \alpha < 1$ and using the first part of (ii) we obtain (3.5) and (3.6).

Define resolvent operators $R(t, \tau)$ and $S(t, \tau)$ by (1.5) and (1.6) respectively. By (3.5) and (3.6) we have

$$\|A^\alpha R(t, \tau)\| < C_\alpha a(t, \tau)(t - \tau)^{-\alpha}, \qquad 0 < \alpha < 1, \tag{3.14}$$

$$\|A^\alpha S(t, \tau)\| < C_\alpha (t - \tau)^{-\alpha}, \qquad 0 < \alpha < 1, \tag{3.15}$$

where $\|\cdot\|$ is the operator norm. It follows easily from the continuity of $r_\lambda(t,\tau)$ and $s_\lambda(t,\tau)$ and the dominated convergence theorem that, for each $x \in H$, $A^\alpha R(t,\tau)x$ and $A^\alpha S(t,\tau)x$ are continuous in (t,τ). Next we shall show that

$$\int_\tau^t a(t,u)AS(u,\tau)x\,du = x - S(t,\tau)x \tag{3.16}$$

and

$$\int_\tau^t a(t,u)AR(u,\tau)x\,du = a(t,\tau)x - R(t,\tau)x \tag{3.17}$$

for $x \in D(A^\alpha)$ and $0 < \alpha < 1$.

To establish (3.16) note that $a(t,u)AS(u,\tau)x = a(t,u)A^{1-\alpha}S(u,\tau)A^\alpha x$ so that by (3.15) the integral in (3.16) makes sense. Hence if $y \in H$ we have

$$\begin{aligned}\left\langle \int_\tau^t a(t,u)AS(u,\tau)x\,du, y\right\rangle &= \int_\tau^t\int_0^\infty \lambda a(t,u)s_\lambda(u,\tau)\,dE_{x,y}(\lambda)\,du\\ &= \int_0^\infty\int_\tau^t \lambda a(t,u)s_\lambda(u,\tau)\,du\,dE_{x,y}(\lambda)\\ &= \int_0^\infty [1 - s_\lambda(t,\tau)]\,dE_{x,y}(\lambda) = \langle x - S(t,\tau)x, y\rangle \end{aligned}\tag{3.18}$$

which establishes (3.16). The proof of (3.17) is similar.

To complete the proof of Theorem 1 we must show that if $x_1(t) = f_1(t) - \int_0^t R(t,\tau)Af_1(\tau)\,d\tau$ and if $x_2(t) = S(t,0)f_2(0) + \int_0^t S(t,\tau)f'(\tau)\,d\tau$ then $x_1(t)$ and $x_2(t)$ satisfy (1.1). If $x_1(t)$ is as given above we have

$$\begin{aligned}\int_0^t a(t,\tau)Ax_1(\tau)\,d\tau &= \int_0^t a(t,\tau)Af_1(\tau)\,d\tau - \int_0^t\int_0^\tau a(t,\tau)AR(\tau,u)Af_1(u)\,du\,d\tau\\ &= \int_0^t a(t,\tau)Af_1(\tau)\,d\tau - \int_0^t\int_u^t a(t,\tau)AR(\tau,u)\,d\tau\,Af_1(u)\,du\\ &= \int_0^t a(t,\tau)Af_1(\tau)\,d\tau - \int_0^t a(t,u)Af_1(u)\,du + \int_0^t R(t,u)Af_1(u)\,du\\ &= f_1(t) - x_1(t)\end{aligned}\tag{3.19}$$

by (3.17) and the fact that $Af_1(u) \in D(A^\alpha)$. Similarly we have

$$\begin{aligned}\int_0^t a(t,\tau)Ax_2(\tau)\,d\tau &= \int_0^t a(t,\tau)AS(\tau,0)f_2(0)\,d\tau\\ &\quad + \int_0^t\int_u^t a(t,\tau)AS(\tau,u)\,d\tau\,f_2'(u)\,du\\ &= f_2(0) - S(t,0)f_2(0) + \int_0^t f_2'(u)\,du - \int_0^t S(t,u)f_2'(u)\,du\\ &= f_2(t) - \left[S(t,0)f_2(0) + \int_0^t S(t,u)f_2'(u)\,du\right]\\ &= f_2(t) - x_2(t)\end{aligned}\tag{3.20}$$

by (3.16) and the fact that $f_2' \in D(A^\alpha)$. Now (2.1) follows immediately from this representation of solutions and uniqueness is proved exactly as in Lemma 2.3 of [1].

The proof of Theorem 2 only entails one major change in the proof of Theorem 1, namely establishing inequalities similar to (3.5) and (3.6) for $\alpha = 0$. We now wish to show that

$$\sup_{\lambda > 0} |r_\lambda(t, \tau)| \leq 2a(\tau, \tau) \tag{3.21}$$

and

$$\sup_{\lambda > 0} |s_\lambda(t, \tau)| \leq 1. \tag{3.22}$$

We begin by replacing t by $t + \tau$ in (1.3) and (1.4). A simple change of variable and the substitutions $\tilde{r}_\lambda(t) = r_\lambda(t + \tau, \tau)$, $\tilde{s}_\lambda(t) = s_\lambda(t + \tau, \tau)$, $\tilde{a}(t) = a(t + \tau, \tau)$ and $b(t, y) = a(t + \tau, y + \tau)$ allow us to rewrite (1.3) and (1.4) as

$$\tilde{r}_\lambda(t) + \int_0^t \lambda b(t, y) \tilde{r}_\lambda(y)\, dy = \tilde{a}(t) \tag{3.23}$$

and

$$\tilde{s}_\lambda(t) + \int_0^t \lambda b(t, y) \tilde{s}_\lambda(y)\, dy = 1. \tag{3.24}$$

The kernel $\lambda b(t, y)$ (τ fixed) satisfies the hypotheses of Theorem 2 of [**8**] and hence we may conclude that

$$|\tilde{r}_\lambda(t)| \leq 2a(\tau, \tau), \tag{3.25}$$

$$|\tilde{s}_\lambda(t)| \leq 1. \tag{3.26}$$

This immediately establishes (3.21) and (3.22). Exactly as before one can show that (3.16) and (3.17) are valid for $x \in D(A)$ and hence the calculations in (3.19) and (3.20) are still valid under the more restrictive hypotheses on $f(t)$.

The uniqueness of the solution is proved as before with the following minor change. If we define $J_n = (I + A/n)^{-1}$ and $A_n = n(I - J_n)$ then A_n is a bounded, positive, selfadjoint operator satisfying $A_n = AJ_n$. If $\{E_n(\lambda) | \lambda \geq 0\}$ is the resolution of the identity associated with A_n, then we have $dE_n(\lambda) = (1 + \lambda/n)^{-1} dE(\lambda)$. Hence, if $R_n(t, \tau)$ is the resolvent operator given by A_n (cf. (1.5)) then $R_n(t, \tau) = J_n R(t, \tau)$. It follows immediately that $\int_0^t R_n(t, \tau) g(\tau)\, d\tau \to \int_0^t R(t, \tau) g(\tau)\, d\tau$ in $L^1[0, T; H]$ for every $g \in L^1[0, T; H]$. This fact, combined with the proof of Lemma 2.3 of [**1**], will establish uniqueness.

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