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OPEN CENTRALIZERS AND THE CONTINUITY OF GROUP REPRESENTATIONS

CLAUDE SCHOCHET AND BERTRAM SCHREIBER[1]

ABSTRACT. Let G be a locally compact group, $\pi\colon G \to \mathcal{L}(L^2(G))$ the right regular representation of G, and $G^c = \{x \in G\colon$ the function $g \rightsquigarrow \pi(gxg^{-1})$ is norm continuous$\}$. This note is devoted to the study of G^c. In particular, the compactly generated groups for which $G = G^c$ are characterized.

1. Let G be a locally compact group and let $\pi\colon G \to \mathcal{L}(L^2(G))$ be the right regular representation of G on $L^2(G)$ with respect to a right Haar measure. The function π is continuous when $\mathcal{L}(L^2(G))$ is given the strong operator topology, but π is not continuous with respect to the norm topology, except in trivial cases. Nevertheless, there is a middle ground, to which this note is devoted.

Following [**5**], [**8**], let $\mathcal{L}_G = \{T \in \mathcal{L}(L^2(G))\colon$ the function $g \rightsquigarrow \pi(g)T\pi(g)^*$ is norm continuous$\}$. Then $\mathcal{L}_G$ is a C^*-algebra which contains the compact operators and has various pleasing properties (cf. [**5**, Theorem 2.2]). Let

$$\begin{aligned} G^c &= \{x \in G\colon \pi(x) \in \mathcal{L}_G\} \\ &= \{x \in G\colon \text{the function } g \rightsquigarrow \pi(gxg^{-1}) \text{ is norm continuous}\}. \end{aligned}$$

It is easy to see that G^c is a subgroup of G. If G is abelian or discrete, then $G^c = G$; in general it is much smaller. The relationship between G^c and G is the main subject of this note.

We shall denote the identity component and the center of G by G_0 and $Z(G)$, respectively. For $x, y \in G$, $C_G(x)$ denotes the centralizer of x in G and $[x, y] = xyx^{-1}y^{-1}$.

DEFINITION 1.1. Let B be a Banach space of functions on G. Suppose that there exist constants $C, \delta > 0$ such that the following conditions are satisfied:

(i) If $\varphi \in B$ and $x \in G$, then ${}_B\pi(x)\varphi \in B$, and $\|{}_B\pi(x)\varphi\| \leqslant C\|\varphi\|$, where ${}_B\pi(x)\varphi(t) = \varphi(tx^{-1})$.

(ii) Given $\varphi \in B$, there exists $\lambda(\varphi) > 0$ such that $\|\varphi + \psi\| \geqslant \lambda(\varphi)$ for all $\psi \in B$ such that $\varphi \cdot \psi = 0$.

(iii) For every neighborhood U of e in G there exists $0 \neq \varphi \in B$ such that $\varphi = 0$ off U and $\lambda(\varphi) \geqslant \delta\|\varphi\|$.

Then B will be called a *homogeneous separating* Banach space of functions on G.

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EXAMPLES 1.2. Most of the Banach spaces commonly encountered in harmonic analysis satisfy the conditions of Definition 1.1. The following is a sampling of such spaces.

(i) $L^p(G)$, $1 \leq p \leq \infty$,

(ii) $C_0(G)$,

(iii) the Fourier algebra $A(G)$ **[1]**,

(iv) the Sobolev spaces $W_k^p(G)$ for G a Lie group,

(v) the spaces $A_p^q(G) = L^p(G) \hat{\otimes} L^q(G)$ of Figà-Talamanca and Gaudry **[2]**, **[3]** on $G \times G$.

DEFINITION 1.3. Let B be a homogeneous separating Banach space of functions on G and ${}_B\pi\colon G \to \mathfrak{L}(B)$ be the regular representation as in (i) of Definition 1.1. Set

$$ {}_BG^c = \left\{ x \in G\colon \text{the function } g \rightsquigarrow {}_B\pi(gxg^{-1}) \text{ is norm continuous} \right\}. $$

The following observation provides the technical tool which is the key to studying ${}_BG^c$.

THEOREM 1.4. *Let B, ${}_B\pi$ and ${}_BG^c$ be as in Definition* 1.3, *and let $x \in G$. Then $x \in {}_BG^c$ if and only if $C_G(x)$ is an open subgroup of G.*

In view of Theorem 1.4 the subscript B in ${}_BG^c$ is redundant and will be omitted following the proof of this theorem. Furthermore, the superscript in G^c may be read as referring to (norm) continuity or to (open) centralizers.

PROOF. Suppose first that $C_G(x)$ is open in G. Then the function $g \rightsquigarrow {}_B\pi(gxg^{-1})$ is constant when restricted to the open subgroup $C_G(x)$ of G, so in particular it is norm continuous at the identity. Since ${}_B\pi$ is a norm-bounded representation of G (1.1(i)), it follows that the function in question is continuous on all of G.

Conversely, suppose that $C_G(x)$ is not an open subgroup of G. Then any neighborhood V of e must contain some $v \in V \setminus C_G(x)$. We shall show that $\|{}_B\pi(gxg^{-1}) - {}_B\pi(x)\|$ is bounded away from zero for all such g, hence $x \notin {}_BG^c$.

As $[g, x] \neq e$ there is a neighborhood W of e such that $W \cap W[g, x] = \varnothing$. Let $0 \neq \varphi \in B$ such that $\varphi = 0$ off W and $\lambda(\varphi) \geq \delta\|\varphi\|$. Then ${}_B\pi([g, x])$ is supported on $W[g, x]$. Thus

$$ \begin{aligned} \|{}_B\pi(gxg^{-1}) - {}_B\pi(x)\| &= \|({}_B\pi([g, x]) - I)\,{}_B\pi(x)\| \geq \|{}_B\pi([g, x]) - I\|/C \\ &\geq \|{}_B\pi([g, x])\varphi - \varphi\|/C\|\varphi\| \geq \lambda(\varphi)/C\|\varphi\| \geq \delta/C. \quad \square \end{aligned} $$

COROLLARY 1.5. *Let G be a connected group. Then $G^c = Z(G)$.*

PROOF. For any group G one has $Z(G) \subset G^c$ trivially. Conversely, suppose that $x \in G^c$. Then $C_G(x)$ is an open subgroup of G. But G is connected, so $C_G(x) = G$. Thus $x \in Z(G)$. $\square$

COROLLARY 1.6. *Suppose that $G^c = G$. Then $G_0 \subset Z(G)$.*

The remainder of this note is devoted to the study of G^c. In §2 we consider the case when $G = G^c$ and we obtain complete information when G is compactly generated. §3 is concerned with other cases.

2. In this section we characterize the compactly generated groups for which $G = G^c$ and we obtain various equivalent formulations of this property.

LEMMA 2.1. *Suppose that $G = G^c$. Let A and B be compact sets of G. Then*

$$[A, B] \equiv \{[x, y]: x \in A, y \in B\}$$

is a finite set.

PROOF. Let $w, x, y, z \in G$ with

$$z \in C_G(x) \cap C_G(y) \tag{2.2}$$

and

$$w \in C_G(x) \cap C_G(y) \cap C_G(z). \tag{2.3}$$

Then

$$\begin{aligned} [zx, wy] &= zxwyx^{-1}z^{-1}y^{-1}w^{-1} \\ &= xyzwz^{-1}w^{-1}x^{-1}y^{-1} \quad \text{by (2.2) and (2.3)} \\ &= xy[z, w]x^{-1}y^{-1} \\ &= [x, y] \quad \text{by (2.3).} \end{aligned}$$

Choose $x \in A$ and $y \in B$. Then there is a neighborhood $U_x \times V_y$ of $(x, y) \in A \times B$ such that $[u, v] = [x, y]$ for all $(u, v) \in U_x \times V_y$. Pick a finite subcover from the open cover $\{U_x \times V_y\}$, and the lemma follows. □

THEOREM 2.4. *Let G be a compactly generated group. Then $G = G^c$ if and only if $Z(G)$ is an open subgroup of G.*

PROOF. As previously noted, $Z(G) \subset C_G(x)$ for all $x \in G$. If $Z(G)$ is open, then every group $C_G(x)$ is open, and hence $G^c = G$.

Conversely, suppose that $G = G^c$. Let K be a compact neighborhood of e which topologically generates G. It suffices to prove that $C_G(K)$ is open, since $C_G(K) = Z(G)$. By Lemma 2.1, there is a neighborhood U of e with $U \subset K$ and $[K, K] \cap U = \{e\}$. Let V be a symmetric neighborhood of e with $[V, V] \subset U$, so that $[V, V] = \{e\}$. Let H be the subgroup of G generated by V. Then H is an abelian subgroup of G, and H is open since V is open. Choose $k_1, \dots, k_n \in K$ such that $K \subset \bigcup_{j=1}^{n} k_j H$. Let L be the open subgroup of G defined by

$$L = H \cap \bigcap_{j=1}^{n} C_G(k_j).$$

If $k \in K$ then $k = k_j y$ for some j and for some $y \in L$. Thus for any $x \in L$, one has $[k, x] = [k_j y, x] = e$, since $L \subset C_G(k_j)$. This shows that $L \subset C_G(K)$ and so $C_G(K)$ is an open subgroup, completing the proof. □

EXAMPLE 2.5. The hypothesis that G be compactly generated in Theorem 2.4 seems essential. We shall exhibit a group G such that $G^c = G$ but $Z(G)$ is not open. Let H_0 be a finite abelian group, and set $H = \prod_{n=1}^{\infty} H_n$, where each H_n is isomorphic to H_0. Let

$$\Sigma = \bigoplus_{n=1}^{\infty} (Z(2))_n,$$

and let $\alpha: \Sigma \to \mathrm{Aut}(H)$ be the isomorphism such that the image of the generator of the jth summand of Σ interchanges the $2j$th and the $(2j+1)$th coordinates of elements of H. Set $G = H \times_\alpha \Sigma$ (semidirect product), where Σ is given the discrete topology. Then G is a locally compact group with open subgroup H. If $(h, \sigma) \in C_G(x)$ for some element x of G, then a direct computation shows that there exists a finite set J of positive integers depending on σ such that if $h'_k = h_k$ for $k \in J$, then $(h', \sigma) \in C_G(x)$. Thus $C_G(x)$ is open in G. Hence $G^c = G$. On the other hand, $Z(G)$ consists only of elements whose H-coordinates are periodic with period two, so $Z(G)$ is not open in G.

THEOREM 2.6. *The following conditions are equivalent for a locally compact group* G.

(a) $G = G^c$.

(b) $C_G(x)$ *is an open subgroup for all* $x \in G$.

(c) $C_G(K)$ *is open for all compact subsets* K *of* G.

(d) $C_G(H)$ *is open for every compactly generated closed subgroup* H *of* G.

(e) $Z(H)$ *is open in* H *for every compactly generated closed subgroup* H *of* G.

PROOF. The implications (a) $\Leftrightarrow$ (b) are the content of Theorem 1.4. The implication (b) $\Rightarrow$ (c) follows from the proof of Theorem 2.4. The facts that (b) $\Leftarrow$ (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) are routine. To see that (e) $\Rightarrow$ (d), recall that if H is a compactly generated closed subgroup of G, then there is an open compactly generated subgroup H' of G containing H. Applying (e) to H', we conclude that $C_G(H)$ is open in G. □

THEOREM 2.7. *Let* G *be a compact group. Then the following conditions are equivalent.*

(a) $G = G^c$.

(b) G *has finite conjugacy classes.*

(c) G *is a central extension of an open abelian subgroup of finite index.*

PROOF. The implication (a) $\Leftrightarrow$ (b) is immediate from Lemma 2.1. If (a) holds, then $Z(G)$ is an open normal subgroup by Theorem 2.4. Since G is compact, any open subgroup must be of finite index, so (a) $\Rightarrow$ (c). (Note that the subgroup may be taken to be $Z(G)$.) Conversely, suppose that H is an abelian subgroup of finite index in G with H central. Then $Z(G)$ is open, so $C_G(x)$ is open for all $x \in G$. □

Condition (c) in Theorem 2.7 is parallel to a condition appearing in an important theorem of C. C. Moore, which we recall. A group G is said to be of *bounded degree* if the dimensions of the irreducible unitary representations of G are bounded [**4**]. The theorem of Moore [**6**] is as follows. *The locally compact group* G *is of bounded degree if and only if* G *is an extension of an open abelian subgroup of finite index.*

We see immediately that any compact group G with $G = G^c$ must be of bounded degree, but that the converse is false. The simplest example was pointed out to us by I. Kaplansky: Take G to be the noncentral extension of the circle group T by the group of order two. Then G is of bounded degree (in fact the irreducible unitary representations of G have dimension at most two), but for appropriate choice x, $C_G(x)$ has four elements. So $G \neq G^c$; in fact $G^c = T$.

3. §3 is devoted to an exploration of how various conditions imposed upon G^c are reflected in the structure of G.

THEOREM 3.1. *G^c is an open subgroup of G if and only if G has an open abelian subgroup.*

PROOF. If H is an open abelian subgroup of G, then $H \subset C_G(x)$ for every $x \in H$. Hence $H \subset G^c$, so G^c is open. Conversely, suppose that G^c is open. Then G^c contains an open compactly generated subgroup H. Theorem 2.4 implies that $Z(H)$ is open in H and hence in G. Thus $Z(H)$ is an open abelian subgroup of G. □

It would be of interest to characterize those groups G with $G^c = \{e\}$. One motivation for our interest in this question is expressed by the following theorem (3.2). Of course, if G is connected, then $G^c = \{e\}$ just means that G has trivial center. A class of totally disconnected groups with $G^c = \{e\}$ is described in Example 3.3.

THEOREM 3.2. *Let π: $G \to \mathfrak{L}(L^2(G))$ be the right regular representation, and for $x \in G$ let Φ_x be the automorphism of $\mathfrak{L}_G$ given by*

$$\Phi_x(T) = \pi(x)T\pi(x)^*.$$

If $x \in G \setminus G^c$, then Φ_x is outer. In particular, if $G^c = \{e\}$, then

$$G \xrightarrow{\Phi} \mathrm{Aut}(\mathfrak{L}_G)/\mathrm{Inn}(\mathfrak{L}_G)$$

is injective.

PROOF. For $x \in G \setminus G^c$ restrict Φ_x to the compact operators $\mathcal{K}$. Then Φ_x is implemented by conjugation by $\pi(x)$, and $\pi(x) \notin \mathfrak{L}_G$. In fact, $\pi(x)$ is the only unitary operator implementing Φ_x. For if U is a unitary operator implementing Φ_x, then $\pi(x)U^{-1}$ would centralize $\mathcal{K}$; hence $\pi(x) = U$. Thus there is no unitary element of $\mathfrak{L}_G$ which implements Φ_x, so Φ_x is outer. □

EXAMPLE 3.3. Consider the group SL(2, Q_p), where Q_p denotes the p-adic number field. Suppose that x is an element of this group with an open centralizer. Then x must commute with all matrices of the form $\begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$ for α, β sufficiently close to zero with $\alpha\beta = 0$, since every neighborhood of I contains such matrices. An easy computation shows that $x = \pm I$, so SL(2, $Q_p)^c = Z(2)$. A similar computation yields PSL(2, $Q_p)^c = \{e\}$.

More generally, let G be a Zariski-connected semisimple affine algebraic group defined over a local field k (of arbitrary characteristic), and assume that G is almost simple and isotropic over k. Let $G(k)$ denote the group of k-rational points of G. Then $G(k)$ has a natural locally compact topology. Let H be a closed cocompact subgroup of $G(k)$. Then every neighborhood of the identity in H is dense in G with respect to the Zariski topology [7, Lemma 2.1]. If $C_H(x)$ is open for some $x \in H$, then $x \in Z(G)$. Thus $H^c = Z(H)$. In particular, if G has trivial center (e.g., G is of adjoint type), then $H^c = \{e\}$.

In all of the examples considered so far in this note, G^c is a closed subgroup of G. In Example 3.4 we show that this is not always the case. Although we do not know how to characterize those groups G for which G^c is closed, Theorem 3.5 below provides a step in that direction.

EXAMPLE 3.4. Let H be a finite group with trivial center, and let $G = \prod_{n=1}^{\infty} H_n$, H_n being isomorphic to H. Then it is easy to see that

$$G^c = \{x = (x_n) \in G: x_n = e \text{ for all but finitely many } n\} = \bigoplus_{1}^{\infty} H_n.$$

In particular, G^c is not closed in G.

THEOREM 3.5. *Let* $Y(G) = \bigcap_{x \in G^c} C_G(x)$.
(1) *If* $Y(G)$ *is open in* G, *then* G^c *is closed.*
(2) *If* G *is metrizable and* G^c *is compact, then* $Y(G)$ *is open.*

PROOF. The first assertion is clear. To prove the second assertion, let $\{H_n\}$ be a decreasing sequence of open subgroups of G such that $G_0 = \bigcap_{n=1}^{\infty} H_n$ and H_n/G_0 is compact for all n. Then every open subgroup of G contains some H_n. For each n let $G_n = \{x \in G^c: C_G(x) \supset H_n\}$. Then $\{G_n\}$ is an increasing sequence of closed subgroups of G whose union is G^c. If G^c is compact, then the Baire Category Theorem implies that G_k is open in G^c for some k, and hence $G_m = G^c$ for some $m \geqslant k$. Thus $H_m \subset Y(G)$. $\square$

COROLLARY 3.6. *If* G_0 *is open in* G, *then* G^c *is closed.*

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DEPARTMENT OF MATHEMATICS, WAYNE STATE UNIVERSITY, DETROIT, MICHIGAN 48202