A CONTINUOUS VERSION OF THE BORSUK-ULAM THEOREM

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Abstract. Let $p: E \to B$ be an $n$-sphere bundle, $q: V \to B$ be an $R^r$-bundle and $f: E \to V$ be a fibre preserving map over a paracompact space $B$. Let $\tilde{p}: \tilde{E} \to B$ be the projectivized bundle obtained from $p$ by the antipodal identification and let $\tilde{A}_f$ be the subset of $\tilde{E}$ consisting of pairs $(e, -e)$ such that $fe = f(-e)$. If the cohomology dimension $d$ of $B$ is finite then the map $(\tilde{p}|\tilde{A}_f)^*: H^d(B; Z_2) \to H^d(\tilde{A}_f; Z_2)$ is injective for a continuous cohomology theory $H^*$. Moreover, if the $j$th Stiefel-Whitney class of $q$ is zero for $1 \leq j \leq r$ then $(\tilde{p}|\tilde{A}_f)^*$ is injective in degrees $i > d - r$. If all the Stiefel-Whitney classes of $q$ are zero then $(\tilde{p}|\tilde{A}_f)^*$ is injective in every degree.

Introduction. The Borsuk-Ulam theorem [1] says that if $f: S^n \to R^n$ is a map then the set $A_f$ of points $x \in S^n$ such that $fx = f(-x)$ is nonempty. Because $A_f$ is symmetric with respect to the antipodal involution, it is more convenient to consider the subset $\tilde{A}_f$ of the real projective $n$-space $P^n$ corresponding to $A_f$ under the antipodal identification.

If a single $S^n$ and an $R^n$ are replaced by continuous families $E \to B$ with fibre $S^n$ and $V \to B$ with fibre $R^n$ over a space $B$, and if $f$ is replaced by a fibre preserving map $f: E \to V$, one may expect the existence of a cross-section of sorts in the set $A_f$ of the real projective $n$-space $P^n$ corresponding to $A_f$ under the antipodal identification.

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A result in this direction in the case when $E$ is the product bundle $E = S^k \times S^n$ and $V$ is a single $R^n$ follows from a theorem proved by J. E. Connett [2]. In this note we are going to consider this question for fibre preserving maps $E \to V$ where $E$ is an $n$-sphere bundle and $V$ is an $n$-dimensional real vector space bundle over a paracompact space $B$. If $B$ is a point, then the theorem proved below reduces to the classical Borsuk-Ulam theorem.

Main result. If $X$ is a space with an involution $t: X \to X$, we denote by $\overline{X}$ the orbit space $X/t$ of $t$. If $p: E \to B$ is a fibre bundle with a fibre preserving involution $t: E \to E$, we write $\overline{p}: \overline{E} \to B$ for the bundle $p/t: E/t \to B$; its fibre is $\overline{X}$, where $X$ is the fibre of $p$. Thus if $p: E \to B$ is an $n$-sphere bundle, then $\overline{p}: \overline{E} \to B$ is the associated real projective $n$-space bundle.
If $E$ is any space with an involution $t: E \to E$ and $f: E \to V$ is a map of $E$ into some space $V$, let $A_f$ denote the set of points $e \in E$ such that $fe = fte$ and let $\tilde{A}_f$ be the image of $A_f$ in $\tilde{E}$.

We are going to use the Alexander-Spanier cohomology theory $H^* \mod 2$. The coefficient group $\mathbb{Z}_2$ will be suppressed from the notation. If $Z$ is a space, $A$ is a subset of $Z$ and $i: A \to Z$ is the inclusion map, then the image of a cohomology class $z \in H^*(Z)$ under the induced homomorphism $i^*: H^*(Z) \to H^*(A)$ will sometimes be denoted by $z|_A$ and called the restriction of $z$ to $A$. We denote by $\dim Z$ the covering dimension of $Z$ and by $d(Z)$ its cohomology dimension, that is, $d(Z) = \text{Sup}\{m: H^m(Z) \neq 0\}$. We have $d(Z) < \dim Z$ if $Z$ is paracompact. If $q: V \to B$ is a vector space bundle over $B$ then the $j$th Stiefel-Whitney class of $q$ is denoted by $w_j(q)$.

We will assume throughout the paper that $B$ is a paracompact space.

**Theorem.** Let $p: E \to B$ be an $n$-sphere bundle with the antipodal involution, let $q: V \to B$ be an $\mathbb{R}^n$-bundle and let $f: E \to V$ be a fibre preserving map over $B$. If $d(B) < d$ and $w_j(q) = 0$ for $1 < j < r$ then the map $(p|_{\tilde{A}_f})^*: H^i(B) \to H^i(\tilde{A}_f)$ is injective for $i > d - r$.

In the following corollaries we specify particular cases of this theorem to illustrate its significance.

**Corollary 1.** If $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ and if $d(B) = d < \infty$, then the map $(p|_{\tilde{A}_f})^*: H^d(B) \to H^d(\tilde{A}_f)$ is injective.

**Corollary 2.** If $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ and if all the Stiefel-Whitney classes of $q$ are zero then the map $(p|_{\tilde{A}_f})^*: H^i(B) \to H^i(\tilde{A}_f)$ is injective for every $i$.

**Corollary 3.** If $B$ is closed manifold and $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ then $\dim A_f = \dim \tilde{A}_f > \dim B$.

In Corollary 3, we have $d = d(B) = \dim B$ and $H^d(B) \neq 0$. On the other hand, $\dim \tilde{A}_f = \dim A_f$ since the orbit map $A_f \to \tilde{A}_f$ is a double covering.

**Proof of the theorem.** If $X$ is any space with a free involution $t: X \to X$, let $u(X)$ denote its characteristic class. It is an element $u(X) \in H^1(X)$, where $X$ is, as usual, the orbit space of $t$. In other words, $u(X)$ is the Stiefel-Whitney class of the double covering $X \to \tilde{X}$. The class $u(S^n)$ of the antipodal involution generates the polynomial ring $H^*(P^n)$ of height $n$.

Let $b \in B$. Then the fibre of $\tilde{p}$ over $b$ is $\tilde{p}^{-1}b = P^n$ and the polynomial ring $H^*(\tilde{p}^{-1}b)$ is generated by $u(p^{-1}b) \in H^1(\tilde{p}^{-1}b)$. The fibre inclusion $p^{-1}b \to E$ is an equivariant map. By the naturality of $u$, the restriction of $u(E) \in H^1(\tilde{E})$ to the fibre $\tilde{p}^{-1}b$ is equal to $u(p^{-1}b)$. By the Leray-Dold-Hirsch theorem [3, p. 229], $H^*(\tilde{E})$ is an $H^*(B)$-module freely generated by the powers $1, u(E), \ldots, u^n(E)$, with
\[ H^*(B) \text{ acting on } H^*(\tilde{E}) \text{ via the cup product. In other words, the map } \]
\[ \bigoplus_{i=0}^{n} H^{m+i}(B) \to H^{m+n}(\tilde{E}), \]
\[ (x_m, x_{m+1}, \ldots, x_{m+n}) \mapsto \sum_{i=0}^{n} (\tilde{p}^* x_{m+i}) \cup u^{n-i}(E) \]
\[ \text{is an isomorphism. This map restricted to } H^m(B) \text{ gives a monomorphism } \]
\[ \iota: H^m(B) \to H^{m+n}(\tilde{E}), \quad x \mapsto (\tilde{p}^* x) \cup u^n(E). \]

Let 0 be the zero section in \( V \) and \( V_0 = V - 0 \). Then the antipodal map is a free involution in \( V_0 \) and the fibre of the bundle \( q_0 = q|V_0: V_0 \to B \) is \( \mathbb{R}^n = \mathbb{R}^n - (0) \). The bundle \( q_0 \) is fibre homotopy equivalent to its \( S^{n-1} \)-bundle and hence \( H^*(\tilde{V}_0) \) is an \( H^*(B) \)-module freely generated by 1, \( u(V_0), \ldots, u^{n-1}(V_0) \). Moreover, \( u^n(V_0) = \sum_{j=1}^{n} (\tilde{g}_0^* w_j) \cup u^{n-j}(V_0) \), where the coefficient \( w_j = w_j(q) \) is the \( j \)th Stiefel-Whitney class of \( q \) [3, p. 232].

Let \( g: E \to V \) be defined by \( ge = fe - f(-e) \). Then \( g \) is equivariant, \( g(-e) = -ge \), \( A_f = A_g = g^{-1}0 \) and the restriction of \( g \) to \( E_0 = E - A_f \) defines an equivariant map \( g_0: E_0 \to V_0 \). By the naturality of \( u \), we have \( \tilde{g}_0^* u(V_0) = u(E_0) \), where \( \tilde{g}_0: \tilde{E}_0 \to \tilde{V}_0 \) is the map of the orbit bundles induced by \( g_0 \) and \( u(E_0) = u(E)|\tilde{E}_0 \). It follows that

\[ u^n(E)|\tilde{E}_0 = \tilde{g}_0^* u^n(V_0) = \sum_{j=1}^{n} \left[ (\tilde{p}^* w_j)|\tilde{E}_0 \right] \cup \left[ u^{n-j}(E)|\tilde{E}_0 \right] \]
\[ = \left[ \sum_{j=1}^{n} \left( \tilde{p}^* w_j \right) \cup u^{n-j}(E) \right]|\tilde{E}_0. \]

To show that \( (\tilde{p}^* A_f)^* \) is a monomorphism in the degrees specified in the theorem, suppose that \( x \in H^i(B) \) with \( i > d - r \) and \( (\tilde{p}^* A_f)^* x = 0 \), i.e., \( (\tilde{p}^* x)|A_f = 0 \). By the continuity of \( H^* \), there is a neighborhood \( U \) of \( A_f \) in \( E \) such that \( (\tilde{p}^* x)|(\bar{U}) = 0 \) (\( \bar{U} \) denotes, as usual, the image of \( U \) in \( \bar{E} \)). Let \( e: \bar{E} \to (\bar{E}, \bar{U}) \) and \( k: E \to (E, \bar{E}_0) \) be the inclusion maps. Since \( (\tilde{p}^* x)|(\bar{U}) = 0 \), then \( \tilde{p}^* x = e^* y \), for some \( y \in H^i(\bar{E}, \bar{U}) \). Let \( v = u^n(E) - \sum_{j=1}^{n} (\tilde{p}^* w_j) \cup u^{n-j}(E) \). Then \( v|\bar{E}_0 = 0 \); hence \( v = k^* z \), for some \( z \in H^n(\bar{E}, \bar{E}_0) \). Since \( (\bar{E}; \bar{U}, \bar{E}_0) \) is an excisive triad, \( e^* y \cup k^* z = y \cup z = 0 \); hence \( 0 = (\tilde{p}^* x) \cup u^n(E) - (\tilde{p}^* x) \cup [\sum_{j=1}^{n} (\tilde{p}^* w_j) \cup u^{n-j}(E)] \). Therefore

\[ (\tilde{p}^* x) \cup u^n(E) = \sum_{j=1}^{n} \tilde{p}^* (x \cup w_j) \cup u^{n-j}(E). \]

Now if \( j < r \) then \( w_j = 0 \) by the assumption. If \( j > r \) then \( \deg(x \cup w_j) = i + j > i + r > d > d(B) \) since \( i > d - r \). Therefore all the coefficients in this polynomial are zero. Hence \( (\tilde{p}^* x) \cup u^n(E) = 0 \). But \( (\tilde{p}^* x) \cup u^n(E) = \iota x \) and \( \iota \) is a monomorphism. Therefore \( x = 0 \) and thus \( (\tilde{p}^* A_f)^* \) is a monomorphism. Q.E.D.

References


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