A CONTINUOUS VERSION OF THE BORSUK-ULAM THEOREM

JAN JAWOROWSKI

Abstract. Let \( p: E \rightarrow B \) be an \( n \)-sphere bundle, \( q: V \rightarrow B \) be an \( \mathbb{R}^n \)-bundle and \( f: E \rightarrow V \) be a fibre preserving map over a paracompact space \( B \). Let \( \tilde{p}: \tilde{E} \rightarrow B \) be the projectivized bundle obtained from \( p \) by the antipodal identification and let \( \tilde{A}_f \) be the subset of \( \tilde{E} \) consisting of pairs \( (e, -e) \) such that \( f(e) = f(-e) \). If the cohomology dimension \( d \) of \( B \) is finite then the map \( (\tilde{p}|_{\tilde{A}_f})^*: H^d(B; \mathbb{Z}) \rightarrow H^d(\tilde{A}_f; \mathbb{Z}) \) is injective for a continuous cohomology theory \( H^* \). Moreover, if the \( j \)-th Stiefel-Whitney class of \( q \) is zero for \( 1 < j < r \) then \((\tilde{p}|_{\tilde{A}_f})^* \) is injective in degrees \( i > d - r \). If all the Stiefel-Whitney classes of \( q \) are zero then \( (\tilde{p}|_{\tilde{A}_f})^* \) is injective in every degree.

Introduction. The Borsuk-Ulam theorem [1] says that if \( f: S^n \rightarrow \mathbb{R}^n \) is a map then the set \( A_f \) of points \( x \in S^n \) such that \( f(x) = f(-x) \) is nonempty. Because \( A_f \) is symmetric with respect to the antipodal involution, it is more convenient to consider the subset \( \tilde{A}_f \) of the real projective \( n \)-space \( P^n \) corresponding to \( A_f \) under the antipodal identification.

If a single \( S^n \) and an \( \mathbb{R}^n \) are replaced by continuous families \( E \rightarrow B \) with fibre \( S^n \) and \( V \rightarrow B \) with fibre \( \mathbb{R}^n \) over a space \( B \), and if \( f \) is replaced by a fibre preserving map \( f: E \rightarrow V \), one may expect the existence of a cross-section of sorts in the set \( A_f \) of pairs \( (e, -e) \) such that \( e \in E \) and \( fe = f(-e) \), at least on an algebraic level.

A result in this direction in the case when \( E \) is the product bundle \( E = S^k \times S^n \) and \( V \) is a single \( \mathbb{R}^n \) follows from a theorem proved by J. E. Connett [2]. In this note we are going to consider this question for fibre preserving maps \( E \rightarrow V \) where \( E \) is an \( n \)-sphere bundle and \( V \) is an \( n \)-dimensional real vector space bundle over a paracompact space \( B \). If \( B \) is a point, then the theorem proved below reduces to the classical Borsuk-Ulam theorem.

Main result. If \( X \) is a space with an involution \( t: X \rightarrow X \), we denote by \( \overline{X} \) the orbit space \( X/t \) of \( t \). If \( p: E \rightarrow B \) is a fibre bundle with a fibre preserving involution \( t: E \rightarrow E \), we write \( \overline{p}: \overline{E} \rightarrow B \) for the bundle \( p/t: E/t \rightarrow B \); its fibre is \( \overline{X} \), where \( X \) is the fibre of \( p \). Thus if \( p: E \rightarrow B \) is an \( n \)-sphere bundle, then \( \overline{p}: \overline{E} \rightarrow B \) is the associated real projective \( n \)-space bundle.
If $E$ is any space with an involution $t: E \to E$ and $f: E \to V$ is a map of $E$ into some space $V$, let $A_f$ denote the set of points $e \in E$ such that $fe = fte$ and let $\bar{A}_f$ be the image of $A_f$ in $\overline{E}$.

We are going to use the Alexander-Spanier cohomology theory $H^\ast \text{mod} 2$. The coefficient group $\mathbb{Z}_2$ will be suppressed from the notation. If $Z$ is a space, $A$ is a subset of $Z$ and $i: A \to Z$ is the inclusion map, then the image of a cohomology class $z \in H^\ast(Z)$ under the induced homomorphism $i^*: H^\ast(Z) \to H^\ast(A)$ will sometimes be denoted by $z|A$ and called the restriction of $z$ to $A$. We denote by $\dim Z$ the covering dimension of $Z$ and by $d(Z)$ its cohomology dimension, that is, $d(Z) = \sup\{m: H^m(Z) \neq 0\}$. We have $d(Z) < \dim Z$ if $Z$ is paracompact. If $q: V \to B$ is a vector space bundle over $B$ then the $j$th Stiefel-Whitney class of $q$ is denoted by $w_j(q)$.

We will assume throughout the paper that $B$ is a paracompact space.

**Theorem.** Let $p: E \to B$ be an $n$-sphere bundle with the antipodal involution, let $q: V \to B$ be an $\mathbb{R}^n$-bundle and let $f: E \to V$ be a fibre preserving map over $B$. If $d(B) < d$ and $w_j(q) = 0$ for $1 \leq j \leq r$ then the map $(\bar{p}\bar{A}_f)^*: H^i(B) \to H^i(\bar{A}_f)$ is injective for $i > d - r$.

In the following corollaries we specify particular cases of this theorem to illustrate its significance.

**Corollary 1.** If $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ and if $d(B) = d < \infty$, then the map $(\bar{p}\bar{A}_f)^*: H^d(B) \to H^d(\bar{A}_f)$ is injective.

**Corollary 2.** If $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ and if all the Stiefel-Whitney classes of $q$ are zero then the map $(\bar{p}\bar{A}_f)^*: H^i(B) \to H^i(\bar{A}_f)$ is injective for every $i$.

**Corollary 3.** If $B$ is closed manifold and $f: E \to V$ is a fibre preserving map of an $n$-sphere bundle $p: E \to B$ with the antipodal involution into an $\mathbb{R}^n$-bundle $q: V \to B$ then $\dim A_f = \dim \bar{A}_f > \dim B$.

In Corollary 3, we have $d = d(B) = \dim B$ and $H^d(B) \neq 0$. On the other hand, $\dim \bar{A}_f = \dim A_f$ since the orbit map $A_f \to \bar{A}_f$ is a double covering.

**Proof of the theorem.** If $X$ is any space with a free involution $t: X \to X$, let $u(X)$ denote its characteristic class. It is an element $u(X) \in H^1(X)$, where $\overline{X}$ is, as usual, the orbit space of $t$. In other words, $u(X)$ is the Stiefel-Whitney class of the double covering $X \to \overline{X}$. The class $u(S^n)$ of the antipodal involution generates the polynomial ring $H^*(\mathbb{P}^n)$ of height $n$.

Let $b \in B$. Then the fibre of $\bar{p}$ over $b$ is $\bar{p}^{-1}b = \mathbb{P}^n$ and the polynomial ring $H^*(\overline{p}^{-1}b)$ is generated by $u(p^{-1}b) \in H^1(\overline{p}^{-1}b)$. The fibre inclusion $p^{-1}b \to E$ is an equivariant map. By the naturality of $u$, the restriction of $u(E) \in H^1(\overline{E})$ to the fibre $\overline{p}^{-1}b$ is equal to $u(p^{-1}b)$. By the Leray-Dold-Hirsch theorem [3, p. 229], $H^*(\overline{E})$ is an $H^*(B)$-module freely generated by the powers 1, $u(E)$, ..., $u^n(E)$, with
$H^*(B)$ acting on $H^*(\overline{E})$ via the cup product. In other words, the map

$$\bigoplus_{i=0}^{n} H^{m+i}(B) \to H^{m+n}(\overline{E}),$$

$$(x_m, x_{m+1}, \ldots, x_{m+n}) \mapsto \sum_{i=0}^{n} (\overline{p}^*x_{m+i}) \cup u^{n-i}(E)$$

is an isomorphism. This map restricted to $H^m(B)$ gives a monomorphism

$$\iota: H^m(B) \to H^m(\overline{E}), \quad x \mapsto (\overline{p}^*x) \cup u^m(E).$$

Let $0$ be the zero section in $V$ and $V_0 = V - 0$. Then the antipodal map is a free involution in $V_0$ and the fibre of the bundle $q_0 = q|V_0: V_0 \to B$ is $\mathbb{R}^n = \mathbb{R}^n - (0)$. The bundle $q_0$ is fibre homotopy equivalent to its $S^{n-1}$-bundle and hence $H^*(\overline{V}_0)$ is an $H^*(B)$-module freely generated by $1, u(V_0), \ldots, u^{n-1}(V_0)$. Moreover, $u^n(V_0) = \sum_{j=1}^{n} (\overline{g}_0^*w_j) \cup u^{n-j}(V_0)$, where the coefficient $w_j = w_j(q)$ is the $j$th Stiefel-Whitney class of $q$ [3, p. 232].

Let $g: E \to V$ be defined by $ge = fe - f(-e)$. Then $g$ is equivariant, $g(-e) = -ge$, $A_f = A_g = g^{-1}0$ and the restriction of $g$ to $E_0 = E - A_f$ defines an equivariant map $g_0: E_0 \to V_0$. By the naturality of $u$, we have $\overline{g}_0^*u(V_0) = u(E_0)$, where $\overline{g}_0: \overline{E}_0 \to \overline{V}_0$ is the map of the orbit bundles induced by $g_0$ and $u(E_0) = u(E)|\overline{E}_0$. It follows that

$$u^n(E)|\overline{E}_0 = \overline{g}_0^*u^n(V_0) = \sum_{j=1}^{n} \left[ (\overline{p}^*w_j)|\overline{E}_0 \right] \cup \left[ u^{n-j}(E)|\overline{E}_0 \right]$$

$$= \left[ \sum_{j=1}^{n} (\overline{p}^*w_j) \cup u^{n-j}(E) \right]|\overline{E}_0.$$

To show that $(\overline{p}|A_f)^*$ is a monomorphism in the degrees specified in the theorem, suppose that $x \in H^i(B)$ with $i > d - r$ and $(\overline{p}|A_f)^*x = 0$, i.e., $(\overline{p}^*x)|A_f = 0$. By the continuity of $H^*$, there is a neighborhood $U$ of $A_f$ in $E$ such that $(\overline{p}^*x)(U) = 0$ ($\overline{U}$ denotes, as usual, the image of $U$ in $\overline{E}$). Let $e: \overline{E} \to (\overline{E}, \overline{U})$ and $k: \overline{E} \to (E, \overline{E}_0)$ be the inclusion maps. Since $(\overline{p}^*x)(U) = 0$, then $\overline{p}^*x = e^*y$, for some $y \in H^i(E, \overline{U})$. Let $\nu = u^n(E) - \sum_{j=1}^{d+r-1} (\overline{p}^*w_j) \cup u^{n-j}(E)$. Then $\nu|\overline{E}_0 = 0$; hence $\nu = k^*z$, for some $z \in H^i(E, \overline{E}_0)$. Since $(\overline{E}; \overline{E}, \overline{E}_0)$ is an excisive triad, $e^*y \cup k^*z = y \cup z = 0$; hence $0 = (\overline{p}^*x) \cup u^n(E) - (\overline{p}^*x) \cup [\sum_{j=1}^{d+r-1} (\overline{p}^*w_j) \cup u^{n-j}(E)]$. Therefore

$$(\overline{p}^*x) \cup u^n(E) = \sum_{j=1}^{n} \overline{p}^*(x \cup w_j) \cup u^{n-j}(E).$$

Now if $j < r$ then $w_j = 0$ by the assumption. If $j > r$ then $\deg(x \cup w_j) = i + j > i + r > d > d(B)$ since $i > d - r$. Therefore all the coefficients in this polynomial are zero. Hence $(\overline{p}^*x) \cup u^n(E) = 0$. But $(\overline{p}^*x) \cup u^n(E) = ix$ and $i$ is a monomorphism. Therefore $x = 0$ and thus $(\overline{p}|A_f)^*$ is a monomorphism. Q.E.D.

REFERENCES


FORSCHUNGSGESELLSCHAFT FÜR MATHEMATIK, ETH ZÜRICH, ZURICH, SWITZERLAND
DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401 (Current address)

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use