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(CA) CLOSURES OF ANALYTIC GROUPS

DAVID ZERLING

ABSTRACT. An analytic group G is called (CA) if the group of inner automorphisms of G is closed in the Lie group of all bicontinuous automorphisms of G. We introduce the notion of a (CA) closure for an analytic group and show that every analytic group possesses a (CA) closure. The definition of uniqueness for such a (CA) closure is developed and a sufficient condition for uniqueness is given.

We also develop new sufficient conditions for a closed normal analytic subgroup of a (CA) analytic group to be (CA).

1. Introduction. By an analytic group and an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup, respectively. If G and H are Lie groups and ϕ is a one-to-one (continuous) homomorphism from G into H, ϕ will be called an immersion. ϕ will be called closed or dense, as $\phi(G)$ is closed or dense in H. G_0 and $Z(G)$ will denote the identity component group and center of G, respectively.

If G is an analytic group, $A(G)$ will denote the Lie group of all (bicontinuous) automorphisms of G, topologized with the generalized compact-open topology. G will be called (CA) if $I(G)$, the Lie group of all inner automorphisms of G, is closed in $A(G)$. It is well known that G is (CA) if and only if its universal covering group is (CA).

If G is a normal analytic subgroup of an analytic group H, then each element h of H induces an automorphism of G, namely, $g \mapsto hgh^{-1}$. We will denote this homomorphism from H into $A(G)$ by ρ_{GH}. $I_H(h)$ will denote the inner automorphism of H determined by $h \in H$. More generally, if A is a subset of H, $I_H(A)$ will denote the set of all inner automorphisms of H determined by elements of A. $I_H(H)$ will be written as $I(H)$, and the mapping $h \mapsto I_H(h)$ of H onto $I(H)$ will be denoted by I_H.

If N is an analytic group and H is an analytic subgroup of $A(N)$, then $N \circledS H$ will denote the semidirect product of N and H. On the other hand, if G is an analytic group containing a closed normal analytic subgroup N and a closed analytic subgroup H, such that $G = NH$, $N \cap H = \{e\}$, and such that the restriction of ρ_{NG} to H is one-to-one, we will frequently identify G with $N \circledS \rho_{NG}(H)$ and H with $\rho_{NG}(H)$, that is, we may write $G = N \circledS H$.

In Zerling [6, The Main Structure Theorem], and [7, Lemma 2.11] the author proved the following theorem:

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THEOREM A. *Let G be a non-*(CA) *analytic group. Then there is a maximal* (CA) *closed normal analytic subgroup M of G, a toral subgroup T in $A(M)$, and a dense vector subgroup V of T, such that:*

(i) *$P = M ⓢ T$ is a* (CA) *analytic group.*

(ii) *G is isomorphic to the dense analytic subgroup $M ⓢ V$ of P.*

(iii) *$Z(G)$ is contained in M.*

(iv) $Z_0(G) = Z_0(P)$.

(v) *Each automorphism σ of G can be extended to an automorphism $\varepsilon(\sigma)$ of P, such that ε: $A(G) \to A(P)$ is a closed immersion.*

In this paper we shall improve upon (iv) by showing that $Z(G) = Z(P)$. We shall develop a sufficient condition for a non-(CA) analytic group to possess a unique (CA) closure, as defined in §2, and we show that each non-(CA) analytic group contains a closed non-(CA) analytic subgroup satisfying this sufficient condition. We shall also develop new sufficient conditions for a closed normal analytic subgroup of a (CA) analytic group to be (CA).

The following results of Goto will be very important to us.

GOTO [**2**, (5.2)]. *Let L be an analytic group and let N be a closed normal analytic subgroup of L. If L/N is a toral group, then there is a toral group T in L such that $L = NT$, $N \cap T = \{e\}$.*

GOTO [**2**, Theorem 2]. *Let G be a dense analytic subgroup of an analytic group L and suppose that G contains a maximal normal analytic subgroup N which contains the commutator subgroup of G and is also closed in L. Then there is a closed vector subgroup V of G, such that $G = NV$, $L = N\overline{V}$, where $N \cap \overline{V} = \{e\}$ and $\overline{V}$ is a toral subgroup of L.*

REMARK. [**2**, (5.2)] and the consequent Theorem 2 of Goto above are generalizations of weaker results in Goto [**1**]. In particular $N \cap \overline{V}$ was only shown to be finite. Since $N \cap \overline{V} = \{e\}$ now, we can improve upon (iv) of Theorem A.

2. Existence and uniqueness of (CA) closures.

DEFINITION. Let G be an analytic group. By a (CA) closure of G we mean a triple (G, f, L), where L is a (CA) analytic group, f: $G \to L$ is a dense immersion, and $Z(f(G)) = Z(L)$.

Let G be a non-(CA) analytic group and let us adopt the notation of Theorem A. Let M' be a maximal analytic subgroup of $I(G)$ which contains the commutator subgroup of $I(G)$ and is closed in $A(G)$. Then from Goto [**2**, Theorem 2] there is a closed vector subgroup V' of $I(G)$, such that $I(G) = M'V'$, $\overline{I(G)} = M'\overline{V}'$, $M' \cap \overline{V}' = \{e\}$, and $T' = \overline{V}'$ is a toral subgroup of $\overline{I(G)}$. In the proof of Theorem A we have $I_G(M) = M'$, $I_G(V) = V'$ and $\rho_{GP}(T) = T'$, where ρ_{GP} is 1-1 on T.

To see that $Z(G) = Z(P)$ we let $(m, \tau) \in Z(P)$ where $m \in M$, $\tau \in T$. Then $\rho_{GP}(m) \cdot \rho_{GP}(\tau) = e$. Therefore $\rho_{GP}(\tau) = e$ and so $\tau = e$. Hence, $Z(G) = Z(P)$. We now have the following theorem.

THEOREM 2.1. *Every analytic group possesses a* (CA) *closure.*

DEFINITION. Let G be an analytic group. Two (CA) closures (G, ψ_1, L_1) and (G, ψ_2, L_2) of G will be called equivalent if there exists an automorphism α of G and an isomorphism β from L_1 onto L_2 so that $\beta \circ \psi_1 = \psi_2 \circ \alpha$. We will say that G possesses a unique (CA) closure if all (CA) closures of G are equivalent.

REMARK. From van Est [4, Theorem 2.2.1] we know that if G is a dense (CA) analytic subgroup of an analytic group L, then $Z(L) = \overline{Z(G)}$, $L = G \cdot \overline{Z(G)}$, and L is also (CA). Hence, it is clear that each (CA) analytic group has a unique (CA) closure, namely itself.

THEOREM 2.2. *Let G be a* (CA) *analytic group and let N and H be a closed normal analytic subgroup and a closed analytic subgroup of G, respectively, such that $G = NH$, $N \cap H = \{e\}$. Let π denote the natural projection of G onto H. If $\pi(Z(G))$ is closed in H, then N is* (CA).

PROOF. Suppose that N is non-(CA). Let N' be a (CA) analytic group containing N as a dense subgroup, where N' is to be constructed according to Theorem A. From Theorem 2.1 we know $Z(N) = Z(N')$. Let ε: $A(N) \to A(N')$ be the extension homomorphism of Theorem A. Let $\beta = \varepsilon \circ \rho_{NG}$. Then the restriction of β to H is a homomorphism of H into $A(N')$, and we let G' denote the semidirect product of N' and H that is determined by β. Then G is dense in G'.

Let $\{(n_k, h_k)\}$ be a sequence of central elements in G converging in G' to (n', h). Since $\pi(Z(G))$ is closed in H, there exists an element $\bar{n}$ in N so that $(\bar{n}, h)$ is in $Z(G)$. Since $n'\bar{n}^{-1} = (n', h) \cdot (\bar{n}, h)^{-1}$, we see that $n'\bar{n}^{-1}$ is in $Z(G') \cap N'$. Therefore, $n'\bar{n}^{-1}$ is in $Z(N')$. Since $Z(N) = Z(N')$, $n'\bar{n}^{-1}$ is in $Z(N)$. Therefore, since $n'\bar{n}^{-1}$ is already in $Z(G')$, it follows that $n'\bar{n}^{-1} \in Z(G)$. So $(n', h) = z \cdot (\bar{n}, h)$, $z \in Z(G)$. So $Z(G)$ is closed in G'. Since G is (CA), $G = G'$ by van Est [4, Theorem 2.2.1]. Hence $N = N'$. Therefore N is (CA). Q.E.D.

COROLLARY. *Let G be a* (CA) *analytic group and let N be a closed normal analytic subgroup of G. If* (i) *$Z(G) \cap N$ is a uniform subgroup of $Z(G)$, and* (ii) *G/N is a toral group, then N is* (CA).

PROOF. From Goto [2] there is a toral group T of G such that $G = N \cdot T$, $N \cap T = \{e\}$. Since $Z(G) \cap N$ is uniform in $Z(G)$, $\pi(Z(G))$ is compact, where π is the natural projection of G onto T. Therefore, N is (CA) from Theorem 2.2. Q.E.D.

THEOREM 2.3. *Adopting the notation of Theorem* A *let $G = MV$ be a non-*(CA) *analytic group. Let f: $G \to L$ be a dense immersion of G into a* (CA) *analytic group L. Suppose $Z(f(G))$ is a uniform subgroup of $Z(L)$. Then there is a closed vector subgroup W of G such that $G = MW$, $L = f(M) \cdot \overline{f(W)}$, $f(M) \cap \overline{f(W)} = \{e\}$, and $\overline{f(W)}$ is a toral group.*

PROOF. From Theorem 2.1 of Zerling [7] we know that $\overline{f(M)} = f(M) \cdot \overline{f(Z(G))}$. Therefore $f(M)$ is closed in L. Let J be a maximal analytic subgroup of G, which contains M and for which $f(J)$ is closed in L. Let π denote the natural projection of J on V. Then $J = MU$, where $U = \pi(J)$.

From Goto [2, Theorem 2] there is a closed vector subgroup W of G so that $G = J \cdot W$, $L = f(J) \cdot \overline{f(W)}$, $f(J) \cap \overline{f(W)} = \{e\}$, and $\overline{f(W)}$ is a toral group. Since $Z(f(G))$ is contained in $f(M)$ and is uniform in $Z(L)$, $\pi'(Z(L))$ is compact in $\overline{f(W)}$, where π' is the natural projection of L onto $\overline{f(W)}$. By Theorem 2.2, J is (CA), since L is (CA). But M is a maximal (CA) closed normal analytic subgroup of G from Theorem A. Therefore, $J = M$. Q.E.D.

THEOREM 2.4. *Let G be a non-*(CA) *analytic group and suppose that $G/Z(G)$ is homeomorphic to Euclidean space. Then G has a unique* (CA) *closure.*

PROOF. Let $f\colon G \to P$ be the dense immersion of Theorem 2.1. That is, $G = MV \cong M \circledS \rho_{MG}(V)$, $P = M \circledS \overline{\rho_{MG}(V)}$, and $f\colon G \to P$ is given by $f(mv) = (m, \rho_{MG}(v))$. From the convention in the Introduction we will write $G = M \circledS V$ and $P = M \circledS \overline{f(V)}$. Let (G, ψ, L) also be a (CA) closure of G. We will show that (G, f, P) is equivalent to (G, ψ, L).

Since $Z(\psi(G)) = Z(L)$, we know from Theorem 2.3 that there is a closed vector subgroup W of G so that $G = MW$, $L = \psi(M) \cdot \overline{\psi(W)}$, $\psi(M) \cap \overline{\psi(W)} = \{e\}$, and $\overline{\psi(W)}$ is a toral group. Since $Z(L) = Z(\psi(G)) \subset \psi(M)$, ρ_{GL} is 1-1 on $\overline{\psi(W)}$. Therefore $L = \psi(M) \circledS \overline{\psi(W)}$.

Let $M' = I_G(M)$, $W' = I_G(W)$, and $V' = I_G(V)$. We see that

$$\overline{I(G)} = M' \cdot \rho_{GP}\overline{(f(V))} = M'\overline{V}', \quad M' \cap \overline{V}' = \{e\}.$$

But $\overline{I(G)} = M' \cdot \rho_{GL}\overline{(\psi(W))} = M' \cdot \overline{W}'$, $M' \cap \overline{W}' = \{e\}$. Therefore, $\overline{V}'$ and $\overline{W}'$ are each maximal toral subgroups of $\overline{I(G)}$. Hence, there is an element γ of $\overline{I(G)}$ so that $\overline{W}' = \gamma\overline{V}'\gamma^{-1}$. But M is γ-stable. Therefore $\gamma|_M \in A(M)$.

Now $\gamma V'\gamma^{-1} \subset I(G) = M'W'$, since $I(G)$ is normal in $\overline{I(G)}$. On the other hand $\gamma V'\gamma^{-1} \subset \overline{W}'$ and $\overline{W}' \cap M' = \{e\}$. Hence $\gamma V'\gamma^{-1} = W'$. Consequently $\gamma|_M \cdot \overline{f(V)} \cdot \gamma|_M^{-1} = \overline{\psi(W)}$ and $\gamma|_M \cdot f(V) \cdot \gamma|_M^{-1} = \psi(W)$.

Let $\beta\colon P \to L$ be given by

$$\beta(m, \tau) = \left(\psi(\gamma(m)), \gamma|_M \cdot \tau \cdot \gamma|_M^{-1}\right), \qquad m \in M, \tau \in \overline{f(V)}.$$

Then β is an isomorphism of P onto L, and since $\rho_{MG}(\gamma(v)) = \gamma|_M \cdot \rho_{MG}(v) \cdot \gamma|_M^{-1}$, we see that $\beta \circ f = \psi \circ \gamma$. Hence, (G, f, P) is equivalent to (G, ψ, L). Q.E.D.

REMARK. Since a semisimple analytic subgroup of the general linear group possesses a nontrivial compact subgroup, we see that the condition of Theorem 2.4 that $G/Z(G)$ is homeomorphic to Euclidean space implies that G is solvable.

3. Abundance of groups with unique (CA) closure.

LEMMA 3.1. *Let $G = MV$ be a non-*(CA) *analytic group as in Theorem* A.

(i) *If $Z(M)$ is connected, then $Z(G)$ is connected.*

(ii) *If $M/Z(M)$ is homeomorphic to Euclidean space, then $G/Z(G)$ is homeomorphic to Euclidean space.*

PROOF. (i) Since $Z(M)$ is a connected abelian group containing $Z(G)$ we let S be a minimal abelian analytic subgroup of G which contains $Z(G)$ and is contained in $Z(M)$. From Goto [2, (7.2) and (8.1)] there is a closed vector subgroup W of G

such that $G = MW$, $M \cap W = \{e\}$, and a closed abelian analytic subgroup H of G (called a *gm*-torus of G) such that H contains both S and W. (The existence of H containing S follows from Goto's (8.1) and the existence of W, for such an H, follows from Goto's (7.2)). Therefore, S commutes with each element of M and W, i.e., $S = Z(G)$. Thus, $Z(G)$ is connected.

(ii) Since $M/Z(M)$ is homeomorphic to Euclidean space, $Z(M)$ and, therefore, $Z(G)$ are connected. Since $M/Z(M) = (M/Z(G))/(Z(M)/Z(G))$, we can show that $M/Z(G)$ is homeomorphic to Euclidean space, if we can show that $Z(M)/Z(G)$ is a vector group.

To this end we will show that $Z(G)$ contains the maximal toral subgroup of $Z(M)$. V acts on $Z(M)$ via $z \mapsto vzv^{-1}$, $z \in Z(M)$, $v \in V$. Let K be the maximal toral subgroup of $Z(M)$. Then $vKv^{-1} = K$ for all $v \in V$. Since V is connected $vkv^{-1} = k$ for all $v \in V$, $k \in K$. Therefore, each $k \in K$ commutes with the elements of M and V, i.e., $K \subset Z(G)$. Hence $M/Z(G)$ is homeomorphic to Euclidean space.

Since $M/Z(G)$ contains all of the maximal compact subgroups of $G/Z(G)$, we see that $G/Z(G)$ is homeomorphic to Euclidean space. Q.E.D.

LEMMA 3.2. *Let N be a nilpotent analytic group and let V be a vector subgroup of $A(N)$, such that $\overline{V}$ is a toral group. Then*

(i) $G = N \circledS V$ *is non-*(CA).

(ii) $\hat{G} = N \circledS \overline{V}$ *is* (CA) *and* $Z(\hat{G}) = Z(G) \subset N$.

(iii) $\overline{I(G)} = I_G(N) \cdot \overline{I_G(V)}$, *where* $\overline{I_G(V)}$ *is a toral group*, $I_G(N) \cap \overline{I_G(V)} = \{e\}$ *and* $I(G) = I_G(N) \cdot I_G(V)$, *where* $I_G(V)$ *is a vector subgroup of* $\overline{I_G(V)}$.

PROOF. Since N is nilpotent and therefore (CA), $I(N)$ is a closed subgroup of $A(N)$, which is homeomorphic to Euclidean space. Hence, $\overline{V} \cap I(N) = \{e\}$. This implies that the center of $N \circledS \overline{V}$ is contained in N. Therefore, $G = N \circledS V$ is dense in $\hat{G} = N \circledS \overline{V}$ with $Z(G) = Z(\hat{G})$. Hence, G is non-(CA) by van Est [4].

Since N is nilpotent, $I_G(N)$ and $I_{\hat{G}}(N)$ are closed in $A(G)$ and $A(\hat{G})$, resp. Therefore, $\hat{G}$ is (CA) since $I_{\hat{G}}(\overline{V})$ is compact. Also $\overline{I(G)} = I_G(N) \cdot \overline{I_G(V)}$, since $\overline{I_G(V)} = I_G(\overline{V})$ is a toral group.

Since $Z(\hat{G}) \subset N$, $I_G(N) \cap \overline{I_G(V)} = \{e\}$, and $I_G(V)$ is a vector subgroup of $\overline{I_G(V)}$. (Consequently, $I_G(N)$ is a maximal analytic subgroup of $I(G)$ which contains the commutator subgroup of $I(G)$ and is closed in $A(G)$.) Q.E.D.

THEOREM 3.1. *Every non-*(CA) *analytic group L contains a closed non-*(CA) *analytic subgroup G such that $G/Z(G)$ is homeomorphic to Euclidean space* (*and therefore G has a unique* (CA) *closure*). *Moreover, if L is solvable, then G is normal in L.*

PROOF. Since L is non-(CA), the radical of L, R, is also non-(CA) from van Est [5, Theorem 2a]. Let $R = MV$ be the decomposition of Theorem A, and let M denote the closure of the commutator subgroup of R. Since R is solvable, N is nilpotent and therefore (CA). From Zerling [6, Theorem 3.2] V acts effectively on N. Since the closure of $I_R(V)$ in $A(R)$ is a toral group, and since N is characteristic in R, we see that the closure of $\rho_{NR}(V)$ in $A(N)$ is a toral group. Hence, Lemma 3.2 shows that $G = NV$ is a non-(CA) closed normal analytic subgroup of R. We want to show that $G = N \circledS V$ is the "$M \circledS V$" type decomposition of Theorem A. However, this is an immediate result of Lemma 3.2.

Hence, since N is nilpotent, $N/Z(N)$ is homeomorphic to Euclidean space. Therefore, $G/Z(G)$ is homeomorphic to Euclidean space from Lemma 3.1. Q.E.D.

REMARK. In Theorem 3.1 we simply wanted to show that every non-(CA) analytic group contains some non-(CA) analytic subgroup possessing a unique (CA) closure. The relationship between G and L actually exists in the relationship between G and R, the radical of L. This relationship is discussed in greater detail (including some open questions) in Stevens [3].

CONJECTURE. In the proof of Theorem 2.4 we were able to show that $\overline{W'} = \gamma\overline{V'}\gamma^{-1}$, $\gamma \in \overline{I(G)}$, only because we knew $\overline{V'}$ and $\overline{W'}$ were each maximal toral subgroups of $\overline{I(G)}$. However, they were assured of being maximal toral subgroups only because M' was homeomorphic to Euclidean space due to our hypothesis that $G/Z(G)$ is homeomorphic to Euclidean space.

The author conjectures that every non-(CA) analytic group possesses a unique (CA) closure. The existence of the above γ and, therefore, of the unique (CA) closure would still be assured without knowing that M' was homeomorphic to Euclidean space, if we were able to prove the following: Let L be an analytic subgroup of $GL(n, \mathbf{R})$ and let M be a closed normal analytic subgroup of $\overline{L}$. Suppose V and W are each closed vector subgroups of L such that (i) $L = MV = MW$, and (ii) $\overline{L} = M\overline{V} = M\overline{W}$, $M \cap \overline{V} = M \cap \overline{W} = \{e\}$, where $\overline{V}$ and $\overline{W}$ are toral groups. Then there exists $\gamma \in \overline{L}$ such that $\gamma\overline{V}\gamma^{-1} = \overline{W}$.

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DEPARTMENT OF MATHEMATICS AND PHYSICS, PHILADELPHIA COLLEGE OF TEXTILES AND SCIENCE, PHILADELPHIA, PENNSYLVANIA 19144