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CUT POINTS OF X AND THE HYPERSPACE OF SUBCONTINUA $C(X)$

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ABSTRACT. Let X be a nondegenerate metric continuum and p_0 a point with $X = X_1 \cup X_2$, $\{p_0\} = X_1 \cap X_2$, X_1 and X_2 continua. Denote by $C(X)$, $C(X_1)$ and $C(X_2)$ the hyperspaces of nonempty subcontinua of X, X_1 and X_2 respectively.

THEOREM. *$C(X)$ is contractible if and only if $C(X_1)$ and $C(X_2)$ are contractible and either X_1 or X_2 is contractible im kleinen at p_0 (a modification of connected im kleinen at p_0).*

THEOREM. *Let X_1 and X_2 satisfy Kelley's condition K. Then $C(X)$ is contractible when and only when either X_1 or X_2 is connected im kleinen at p_0.*

Examples are given.

Let X be a nondegenerate metric continuum and p_0 be a cut point of X. Denote by X_1 and X_2 subcontinua of X such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{p_0\}$. Each of X, X_1 and X_2 have their respective hyperspaces of nonempty subcontinua $C(X)$, $C(X_1)$ and $C(X_2)$ endowed with the Hausdorff metric D. In the present paper, a characterization of the contractibility of $C(X)$ is proved in terms of properties of the subcontinua X_1 and X_2. A corollary is then proved in which a characterization of the contractibility of $C(X)$ is established when both X_1 and X_2 have the property K of **[2]**. Further applications of the main characterization theorem are also given.

Throughout the paper the symbol I will be reserved for the closed interval [0, 1]. For a general reference on $C(X)$, see **[4]**.

1. The fibers of the cut point. By the fibers of the cut point p_0 we will mean the following closed subsets of $C(X)$.

$$\mathfrak{F} = \{A \in C(X) | p_0 \in A\},$$
$$\mathfrak{F}_i = \{A \in C(X_i) | p_0 \in A\} \qquad (i = 1, 2).$$

1.1. PROPOSITION. *$\mathfrak{F}$ and $\mathfrak{F}_1 \times \mathfrak{F}_2$ are homeomorphic. Hence $C(X) = C(X_1) \cup (\mathfrak{F}_1 \times \mathfrak{F}_2) \cup C(X_2)$ with the natural identifications.*

PROOF. Let Φ: $\mathfrak{F}_1 \times \mathfrak{F}_2 \to 2^X$ (2^X = space of nonempty closed subsets of X) be given by $\Phi(A_1, A_2) = A_1 \cup A_2$. Clearly, Φ is continuous and into $\mathfrak{F}$. If $A \in \mathfrak{F}$ then $A_1 = A \cap X_1$ and $A_2 = A \cap X_2$ are connected because p_0 is a cut point of X.

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Hence Φ is onto $\mathfrak{F}$. To show Φ is one-to-one, let (A_1, A_2) and (B_1, B_2) be distinct elements of $\mathfrak{F}_1 \times \mathfrak{F}_2$. We may suppose $A_1 \neq B_1$. Since $A_1 \cap B_2 = \{p_0\}$ and $B_1 \cap A_2 = \{p_0\}$, we have $A_1 \cup A_2 \neq B_1 \cup B_2$. We now conclude Φ is a homeomorphism of $\mathfrak{F}_1 \times \mathfrak{F}_2$ onto $\mathfrak{F}$.

2. The first necesary condition. A contraction h: $C(X) \times I \to C(X)$ is called *monotone* if $h(A, t) \subset h(A, t')$ for $A \in C(X)$ and $t < t'$.

2.1. PROPOSITION [2]. *If $C(X)$ is contractible then there is a monotone contraction h: $C(X) \times I \to C(X)$.*

PROOF. Let $\bar{h}$: $C(X) \times I \to C(X)$ be a contraction. Since $C(X)$ is arcwise connected, we may assume $\bar{h}(A, 1) = X$ for all $A \in C(X)$. For each $(A, t) \in C(X) \times I$, define $h(A, t) = \bigcup \{\bar{h}(A, t'): 0 \leqslant t' \leqslant t\}$. Then h has the required property and is a contraction.

2.2. THEOREM. *If $C(X)$ is contractible then so are $C(X_1)$ and $C(X_2)$.*

PROOF. Define γ: $X \to X_i$ ($i = 1$ or 2) by

$$\gamma(x) = \begin{cases} x, & \text{if } x \in X_i, \\ p_0, & \text{otherwise} \end{cases}$$

and define $\bar{\gamma}$: $C(X) \to C(X_i)$ by $\bar{\gamma}(A) = \gamma[A]$ for all $A \in C(X)$. Then, $\bar{\gamma}$ is a retraction from $C(X)$ onto $C(X_i)$. Hence, since $C(X)$ is contractible, so is $C(X_i)$. This proof is the referee's and is somewhat different from our original proof.

3. Contractibility im kleinen. In order to discuss the second necessary condition we must define a notion related to connected im kleinen. (See [1] or [6] for the definition of connected im kleinen. The second reference uses the name locally connected.)

3.1. DEFINITION. Let Y be a metric space and $q_0 \in Y$. We say Y is *contractible im kleinen at q_0* if for each $\varepsilon > 0$ there are $\delta > 0$ and continuous map h_ε: $Y \times I \to C(Y)$ such that

(1) $h_\varepsilon(q, 0) = \{q\}$ for each $q \in Y$;
(2) $d(q, q_0) < \delta$ implies $q_0 \in h_\varepsilon(q, 1)$;
(3) $d(q, q_0) < \delta$ and $t \in I$ imply diam $h_\varepsilon(q, t) \leqslant \varepsilon$; and
(4) $h_\varepsilon(q, t) \subset h_\varepsilon(q, t')$ for $q \in Y$ and $t < t'$.

3.2. PROPOSITION. *If Y is contractible im kleinen at q_0 then Y is connected im kleinen at q_0.*

3.3. DEFINITION [2]. A metric space Y is said to have *property K* if for each $\varepsilon > 0$ there is $\delta > 0$ such that whenever $a \in Y$, $a \in A \in C(Y)$ and $b \in Y$ with $d(a, b) < \delta$ there is $B \in C(Y)$ with $b \in B$ and $D(A, B) < \varepsilon$.

It is proved in [2] that property K implies $C(Y)$ is contractible for a continuum Y.

3.4. THEOREM. *If a continuum Y has property K and is connected im kleinen at q_0 then it is contractible im kleinen at q_0.*

PROOF. As in [2], there is a continuous function μ: $C(Y) \to I$ such that $\mu(A) = 0$ if and only if A is a one-point set and $\mu(A) < \mu(B)$ if $A \subset B$ and $A \neq B$. It is proved in [2, pp. 23–24] with the aid of property K, that the map h: $Y \times I \to C(Y)$, given by $h(q, t) = \bigcup \{A | q \in A \text{ and } \mu(A) \leqslant t\}$, is continuous. Let $\varepsilon > 0$ be given. Then the set

$$W = \{A \in C(Y) | \exists q_1, q_2 \in A \ni d(q_1, q_0) \geqslant \varepsilon/2 \text{ and } d(q_2, q_0) \leqslant \varepsilon/4\}$$

is compact. Let $2\lambda = \min\{\mu(A) | A \in W\}$. Then $\lambda > 0$. Since Y is connected im kleinen at q_0, there is a continuum U with q_0 as an interior point such that $\mu(U) < \lambda$. Let $\delta > 0$ be such that $d(q, q_0) < \delta$ implies $q \in U$. We may assume $\delta < \varepsilon/4$. Then for $d(q, q_0) < \delta$ we have $q \in A \in C(Y)$ and $\mu(A) \leqslant \lambda$ implies $\operatorname{diam} A \leqslant \varepsilon$. Hence for $d(q, q_0) < \delta$ we have $q_0 \in h(q, \lambda)$ and $\operatorname{diam} h(q, \lambda) \leqslant \varepsilon$. The proof is now easily completed.

3.5. PROPOSITION. *Let Y be a continuum which is contractible im kleinen at q_0. Then for each $\varepsilon > 0$ there are $\delta > 0$ and continuous map S_ε: $Y \times I \to C(Y)$ such that*

(1) $S_\varepsilon(q, 0) = \{q\}$ *for* $q \in Y$;
(2) $q_0 \in S_\varepsilon(q, 1)$ *for* $d(q, q_0) < \delta/2$;
(3) $\operatorname{diam} S_\varepsilon(q, t) \leqslant \varepsilon$ *for* $(q, t) \in Y \times I$;
(4) $S_\varepsilon(q, t) = \{q\}$ *for* $d(q, q_0) \geqslant \delta$ *and* $t \in I$; *and*
(5) $S_\varepsilon(q, t) \subset S_\varepsilon(q, t')$ *for* $q \in Y$ *and* $t < t'$.

PROOF. Let $\varepsilon > 0$. Then let δ and h_ε be as in Definition 3.1. Define τ: $Y \to \mathbf{R}$ as the continuous function

$$\tau(q) = \begin{cases} 1, & d(q, q_0) \leqslant \delta/2, \\ 0, & d(q, q_0) \geqslant \delta, \\ 2\delta^{-1}(\delta - d(q, q_0)), & \delta \geqslant d(q, q_0) \geqslant \delta/2. \end{cases}$$

Define S_ε: $Y \times I \to C(Y)$ by

$$S_\varepsilon(q, t) = \begin{cases} h_\varepsilon(q, t), & 0 \leqslant t \leqslant \tau(q), \\ h_\varepsilon(q, \tau(q)), & \tau(q) \leqslant t \leqslant 1. \end{cases}$$

One verifies easily that S_ε has the required properties.

3.6. PROPOSITION. *Let Y be a continuum which is contractible im kleinen at q_0 and $\mathcal{G} = \{A \in C(Y) | q_0 \in A\}$ be the fiber of q_0. Suppose H: $C(Y) \times I \to C(Y)$ is a continuous map such that $H(A, 0) = A$, $H(A, t) \subset H(A, t')$ for $t < t'$ and $H(A, 1) \in \mathcal{G}$ for $A \in C(Y)$. Then for each $\varepsilon > 0$ there are $\eta > 0$ and continuous map H_ε: $C(Y) \times I \to C(Y)$ such that*

(1) $0 < \eta < \varepsilon$;
(2) $H_\varepsilon(A, 0) = A$ *for* $A \in C(Y)$;
(3) $H_\varepsilon(A, t) \subset H_\varepsilon(A, t')$ *for* $A \in C(Y)$ *and* $t < t'$;
(4) $H_\varepsilon(A, 1) \in \mathcal{G}$ *for* $A \in C(Y)$;
(5) $D(H_\varepsilon(A, t), H_\varepsilon(A, 1 - \eta)) < \varepsilon$ *for* $D(A, \mathcal{G}) \geqslant 2\eta$ *and* $1 - \eta \leqslant t$;
(6) $H_\varepsilon(A, t) = H(A, t)$ *for* $D(A, \mathcal{G}) \geqslant 2\eta$ *and* $1 - \eta \geqslant t$;

(7) $D(H_\varepsilon(A, t), A) < \varepsilon$ *for* $D(A, \mathcal{G}) < \eta$ *and* $t \in I$;

(8) $D(H_\varepsilon(A, t), H_\varepsilon(A, \alpha(A))) < \varepsilon$ *for* $\eta < D(A, \mathcal{G}) < 2\eta$ *and* $1 > t > \alpha(A) = [(1 - \eta)]/\eta(D(A, \mathcal{G}) - \eta)$; *and*

(9) $H_\varepsilon(A, t) = H(A, t)$ *for* $\eta < D(A, \mathcal{G}) < 2\eta$ *and* $0 < t < \alpha(A) = [(1 - \eta)]/\eta(D(A, \mathcal{G}) - \eta)$.

PROOF. Let $\varepsilon > 0$ be given and let δ and S_ε be given by Proposition 3.5. The set

$$U = \{(A, t) \in C(Y) \times I \mid d(q_0, H(A, t)) < \delta/2\}$$

is open and contains $C(Y) \times \{1\} \cup \mathcal{G} \times I$, where $\mathcal{G}$ is the fiber of q_0. Let η be such that

(i) $0 < \eta < \varepsilon$;

(ii) $D(A, \mathcal{G}) < \eta$ implies $(A, t) \in U$ for all $t \in I$; and

(iii) $1 - 2\eta < t$ implies $(A, t) \in U$ for all $A \in C(Y)$.

Then the function

$$\alpha(A) = \begin{cases} 1 - \eta, & D(A, \mathcal{G}) > 2\eta, \\ 0, & D(A, \mathcal{G}) < \eta, \\ \dfrac{1 - \eta}{\eta}(D(A, \mathcal{G}) - \eta), & \eta < D(A, \mathcal{G}) < 2\eta, \end{cases}$$

is continuous and $W = \{(A, t) \in C(Y) \times I \mid \alpha(A) < t < 1\} \subset U$. For $\alpha(A) < t < 1$, let

$$\beta(A, t) = (t - \alpha(A))/(1 - \alpha(A))$$

and

$$H(A, \alpha(A))_t = \bigcup \{S_\varepsilon(q, \beta(A, t)) \mid q \in H(A, \alpha(A))\}.$$

Then $\beta(A, t)$ and $H(A, \alpha(A))_t$ are continuous on the closed set W. Let H_ε: $C(Y) \times I \to C(Y)$ be defined by

$$H_\varepsilon(A, t) = \begin{cases} H(A, t), & 0 < t < \alpha(A), \\ H(A, \alpha(A))_t, & \alpha(A) < t < 1, \end{cases}$$

where $A \in C(Y)$. Clearly, H_ε is continuous. Conditions (1)–(9) are easily verified.

3.7. LEMMA. *If* $C(Y)$ *is contractible and* Y *is contractible im kleinen at* q_0 *then there is a deformation retract of* $C(Y)$ *onto the fiber* $\mathcal{G}$ *of* q_0.

PROOF. Let H_0: $C(Y) \times I \to C(Y)$ be a monotone contraction of $C(Y)$ and $0 < \varepsilon_1 < 2^{-1}$. Then there are η_1 and H_{ε_1} from Proposition 3.6. Proceeding inductively, we have sequences $\{\varepsilon_n\}$, $\{\eta_n\}$ and $\{H_{\varepsilon_n}\}$ such that $0 < \varepsilon_n < 2^{-n}$ and η_{n+1} and $H_{\varepsilon_{n+1}}$ are related to ε_n and H_{ε_n} as in Proposition 3.6. Moreover, we may assume $\eta_n > \varepsilon_{n+1}$. One easily sees that the sequence $\{H_{\varepsilon_n}\}$ converges uniformly on $C(Y) \times I$, hence its limit H is continuous. Also, it is easily verified that $H(A, 1) \in \mathcal{G}$ for $A \in C(Y)$ and $H(A, t) = A$ for $A \in \mathcal{G}$ and $t \in I$. An added consequence is $H(A, t) \subset H(A, t')$ for $t < t'$.

4. The second necessary condition. We return to $X = X_1 \cup X_2$ with $X_1 \cap X_2 = \{p_0\}$ as in §§2 and 3 above. Suppose h: $C(X) \times I \to C(X)$ is a monotone contraction. Then $h(\{p_0\}, \cdot)$ maps I into $\mathfrak{F}_1 \times \mathfrak{F}_2$ and $h(\{p_0\}, 0) = (\{p_0\}, \{p_0\})$ and $h(\{p_0\}, 1) = (X_1, X_2)$. If P_i: $\mathfrak{F}_1 \times \mathfrak{F}_2 \to \mathfrak{F}_i$ $(i = 1, 2)$ are the natural projections then the functions

$$s_i(t) = \operatorname{diam} P_i(h(\{p_0\}, t)) \qquad (i = 1, 2)$$

are continuous increasing functions with $s_i(0) = 0$ and $s_i(1) > 0$ $(i = 1, 2)$. Let $t_i = \min\{t | s_i(t) > 0\}$ $(i = 1, 2)$.

4.1. PROPOSITION. *Suppose $t_2 \leqslant t_1$. Then for each $\varepsilon > 0$ there are $\delta > 0$ and $\bar{t}$ such that*

(1) *$d(p, p_0) < \delta, p \in X_1$ and $0 \leqslant t \leqslant \bar{t}$ imply* $\operatorname{diam} h(\{p\}, t) < \varepsilon$; *and*

(2) *$d(p, p_0) < \delta$ and $p \in X_1$ imply $p_0 \in h(\{p\}, \bar{t})$.*

PROOF. There is $\bar{t}$ such that $0 < \operatorname{diam} h(\{p_0\}, \bar{t}) < \varepsilon/2$. Let $A_i = X_i \cap h(\{p_0\}, \bar{t})$ $(i = 1, 2)$. Then $\bar{t} > t_2$ and $\operatorname{diam} A_2 > 0$. Let $\bar{p} \in A_2 \setminus \{p_0\}$ and η be a positive number smaller than $d(\bar{p}, X_1)$. From the uniform continuity of h there is $\delta > 0$ such that $d(p, p_0) < \delta$ implies $D(h(\{p\}, t), h(\{p_0\}, t)) < \min\{\eta, \varepsilon/4\}$ for $t \in I$. So, when $p \in X_1$ with $0 < d(p, p_0) < \delta$ we have $h(\{p\}, \bar{t}) \setminus X_1 \neq \varnothing$ and $h(\{p\}, \bar{t}) \setminus X_2 \neq \varnothing$. Hence $p_0 \in h(\{p\}, \bar{t})$ for $d(p, p_0) < \delta$ and $p \in X_1$ and thereby (2) is proved. Finally for $p \in X_1$, $d(p, p_0) < \delta$ and $0 \leqslant t \leqslant \bar{t}$ we have $\operatorname{diam} h(\{p\}, t) \leqslant \operatorname{diam} h(\{p_0\}, t) + 2D(h(\{p\}, t), h(\{p_0\}, t)) \leqslant \operatorname{diam} h(\{p_0\}, \bar{t}) + \varepsilon/2 < \varepsilon$ and (1) is proved.

4.2. PROPOSITION. *If $t_2 \leqslant t_1$ then X_1 is contractible im kleinen at p_0.*

PROOF. Let $\bar{t}$ and δ be as in Proposition 4.1. Define h_ε: $X_1 \times I \to C(X_1)$ by

$$h_\varepsilon(p, t) = h(\{p\}, t\bar{t}) \cap X_1.$$

4.3. THEOREM. *If $C(X)$ is contractible then either X_1 or X_2 is contractible im kleinen at p_0.*

5. The characterization theorem.

5.1. THEOREM. *$C(X)$ is contractible if and only if $C(X_1)$ and $C(X_2)$ are both contractible and either X_1 or X_2 is contractible im kleinen at p_0.*

PROOF. We need only prove sufficiency. Suppose X_1 is contractible im kleinen at p_0. By Lemma 3.7, there is a deformation retract h_0: $C(X_1) \times I \to C(X_1)$ of $C(X_1)$ onto $\mathfrak{F}_1$. Denote by h_1 and h_2 monotone contractions of $C(X_1)$ and $C(X_2)$. Define h: $C(X) \times I \to C(X)$ by first deforming $C(X) = C(X_1) \cup (\mathfrak{F}_1 \times \mathfrak{F}_2) \cup C(X_2)$ onto $(\mathfrak{F}_1 \times \mathfrak{F}_2) \cup C(X_2)$ by means of h_0; second, deforming $(\mathfrak{F}_1 \times \mathfrak{F}_2) \cup C(X_2)$ onto $\mathfrak{F}_1 \times \{X_2\}$ by means of h_2; finally, deforming $\mathfrak{F}_1 \times \{X_2\}$ to $\{X_1\} \times \{X_2\} = X$ by means of h_1.

5.2. THEOREM. *Suppose X_1 and X_2 both have property K. Then $C(X)$ is contractible when and only when either X_1 or X_2 is connected im kleinen at p_0.*

PROOF. The theorem follows from Theorems 3.4 and 5.1.

5.3. EXAMPLE. We give the example of [2]. Let Y be the closure in the plane of $\{(u, v)|v = \sin(1/u)$ for some $0 < u \leq 1\}$. One can easily show Y has property K. Let X_1 and X_2 be two copies of Y and $X = X_1 \cup X_2$ with $\{p_0\} = X_1 \cap X_2$ such that neither X_1 nor X_2 is connected im kleinen at p_0. Then $C(X)$ is not contractible, a fact already observed by Kelley in [2].

We have the following two consequences of Theorem 5.1.

5.4. PROPOSITION. *If $C(X_1)$ and $C(X_2)$ are contractible and there are arbitrarily small subcontinua X_ε of X_1 whose interior relative to X_1 contains p_0 and $C(X_\varepsilon)$ are contractible, then $C(X)$ is contractible.*

5.5. PROPOSITION. *If $C(X_1)$ and $C(X_2)$ are contractible and X_1 has arbitrarily small contractible subcontinua X_ε whose interiors relative to X_1 contains p_0, then $C(X)$ is contractible.*

5.6. EXAMPLE. Let X_1 be the arc I and X_2 be a pseudo arc such that $X_1 \cap X_2 = \{p_0\}$ and $X = X_1 \cup X_2$. Then $C(X)$ is contractible.

PROOF. By [5], $C(X_2)$ is contractible. See also [3].

6. Remark. There is a continuum Y which is connected im kleinen at q_0 but not contractible im kleinen. Let Y_n be the closure in the plane of the set $\{(u, v + 2n)|v = \sin(1/u)$ for some $0 < u \leq 1\}$ and let Y be the one-point compactification of $\bigcup_{n=0}^{\infty} Y_n$. Then one sees that Y is connected im kleinen at ∞, but not contractible im kleinen at ∞. The latter fact can be proved by using techniques from §4 and the fact that Y is not connected im kleinen at each cut point of Y whose first coordinate is zero.

REFERENCES

1. J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.

2. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 22–36.

3. J. Krasinkiewicz, *On the hyperspaces of hereditarily indecomposable continua*, Fund. Math. **84** (1974), 175–186.

4. S. B. Nadler, Jr., *Hyperspaces of sets*, Dekker, New York, 1978.

5. C. J. Rhee, *On dimension of hyperspace of a metric continuum*, Bull. Soc. Roy. Sci. Liege **38** (1969), 602–604.

6. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, R.I., 1942.

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