PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 2, June 1981

THE ZERO DIVISOR PROBLEM FOR A CLASS OF TORSION-FREE GROUPS

A. I. LICHTMAN

ABSTRACT. Let G be a group, N be a central torsion-free subgroup of G and let R be an arbitrary field. Then $R(G)$ contains no nilpotent elements provided that $R(G/N)$ contains no nilpotent elements.

When G is torsion-free the conditions of the theorem imply that RG is a domain; this generalizes Passman's theorem in [1].

1. Introduction. Let G be a torsion-free group, N a central subgroup of G such that the quotient group G/N is torsion-free. D. Passman proved in [1] that if R is a field of characteristic zero then the group ring RG contains no zero divisors provided that $R(G/N)$ contains none.

Our goal is to establish the truth of this result for a field of arbitrary characteristic. This will follow from the Corollary of the following theorem.

THEOREM 1. *Let G be a group, N be a central torsion-free subgroup of G and let R be an arbitrary field. Assume that $R(G/N)$ contains no nilpotent elements. Then $R(G)$ also does not contain any nilpotent elements.*

When G is torsion-free it is known that RG is a (noncommutative) domain if it contains no nilpotent elements (see [2, 13.1]). We obtain thus the following Corollary of Theorem 1.

COROLLARY. *Let G be a torsion free group, N be a central subgroup of G. Assume that $R(G/N)$ contains no nilpotent elements. Then RG is a domain.*

It is worth remarking that neither in Theorem 1 nor in the Corollary do we assume that G/N is a torsion-free group.

2. Preliminaries.

LEMMA 1. *Let H be a torsion-free abelian group and let $h_1, h_2, \ldots, h_n$ be nonunit elements of H. Then there exists a homomorphism φ of H into the additive group $(Q, +)$ of rationals such that*

$$\varphi(h_j) \neq 0 \qquad (j = 1, 2, \ldots, n). \tag{1}$$

PROOF. The group H can be embedded in a complete abelian group V, which can be considered as a vector space over rationals. Thus, it is enough to prove that there exists a linear functional on V which does not vanish on given nonzero

Received by the editors June 9, 1980.
AMS (MOS) subject classifications (1970). Primary 16A26.
Key words and phrases. Groups, group rings.

elements $h_1, h_2, \ldots, h_n$. When V is a finite-dimensional vector space over a field of zero characteristic this is a well-known fact from linear algebra (see [3, §3.5, Exercise 14]); the infinite-dimensional case is easily reduced to the finite-dimensional one.

We need the concept of discrimination by a family of groups (see [4]):

A family G_i $(i \in I)$ of groups discriminates a group G if for every set $g_1, g_2, \ldots, g_n$ of nontrivial elements of G there exists an $i \in I$ and a homomorphism f of G into G_i such that $f(g_k) \neq 1$ $(k = 1, 2, \ldots, n)$.

Using the concept of discrimination we can formulate Lemma 1 in the following way:

Let H be a torsion-free abelian group. Then H can be discriminated by subgroups of the additive group of rationals.

The following fact is well known (and can be verified easily).

LEMMA 2. *Let G be discriminated by a family of groups G_i $(i \in I)$. Then the group ring RG is a subdirect sum of group rings RG_i $(i \in I)$.*

LEMMA 3. *Let G be a group and N be a central subgroup of G. Assume that N is discriminated by a family of groups N_i $(i \in I)$. Then G is discriminated by a family of groups G_i $(i \in I)$ such that for any $i \in I$, G_i contains a central subgroup $\tilde{N}_i \simeq N_i$ and $G_i/\tilde{N}_i \simeq G/N$.*

The proof is immediate.

3. Proof of Theorem 1. Without loss of generality we can assume that N is locally cyclic. Indeed, Lemmas 1 and 3 show that G is discriminated by a family of groups G_i $(i \in I)$, such that everyone of them contains a central locally cyclic torsion-free subgroup N_i and $G_i/N_i \simeq G/N$; Lemma 2 now implies that RG is a subdirect sum of group rings RG_i $(i \in I)$. Clearly, if any RG_i $(i \in I)$ contains no nilpotent element, then RG also does not contain any nilpotent elements.

Assume therefore that G contains a locally cyclic torsion-free group N such that $R(G/N)$ contains no nilpotent elements, but there exists $0 \neq x \in RG$ such that $x^2 = 0$.

Let g_j $(j \in J)$ be a transversal of N in G and

$$x = \sum_{j \in J_1} \lambda_j g_j, \qquad 0 \neq \lambda_j \in RN \quad (j \in J_1), \tag{2}$$

where J_1 is a finite subset of J.

Let F be an infinite cyclic subgroup of N such that RF contains all the elements λ_j $(j \in J_1)$ in (2) and let f_α $(\alpha \in U)$ be a transversal of F in N. Clearly, $f_\alpha g_j$ $(\alpha \in U, j \in J)$ is a transversal of F in G.

Since F is infinite cyclic, the augmentation ideal $w(RF)$ is principal generated by the element $(f - 1)$, and $\bigcap_{n=0}^{\infty} (f-1)^n RF = 0$. Then, if $(f-1)^k$, $k > 0$, is the maximal power of $(f - 1)$ which divides in RF all the elements λ_j $(j \in J_1)$ in (2), we can write

$$x = (f-1)^k x_1, \tag{3}$$

where

$$x_1 = \sum_{j \in J_1} \mu_j g_j \in RG, \qquad 0 \neq \mu_j \in RF \quad (j \in J_1), \tag{4}$$

and there exists at least one j_0 such that $\mu_{j_0} \notin w(RF)$. The relation $w(RN) \cap RF = w(RF)$ implies then that $\mu_{j_0} \notin w(RN)$.

The relation $x^2 = 0$ now gives, via the fact that $(f - 1)$ is central,

$$(f - 1)^{2k} x_1^2 = 0. \tag{5}$$

It is well known that the element $(f - 1)$ is not a zero divisor in RG if f has an infinite order. We obtain therefore from (5) $x_1^2 = 0$.

Let $\bar{X}$ be the image of a subset $X \subseteq G$ under the natural homomorphism $G \to G/N$. Since $\mu_{j_0} \notin w(RN)$ we obtain that $\bar{\mu}_{j_0} \neq \bar{0}$. On the other hand, the elements g_j $(j \in J_1)$ belong to different cosets in G/N. Hence the element $\bar{x}_1 = \sum_{j \in J_1} \bar{\mu}_j \bar{g}_j$ is a nonzero element in the group ring $R\bar{G}$. We have also

$$\bar{x}_1^2 = \bar{0}, \tag{6}$$

which contradicts the assumption that $R(G/N)$ has no nilpotent elements. □

ACKNOWLEDGEMENT. I am grateful to the referee, who suggested an essential shortening of the proof.

REFERENCES

1. D. Passman, *Observations on group rings*, Comm. Algebra **5** (1977), 1119–1162.
2. ______, *The algebraic structure of group rings*, Pure and Appl. Math., Interscience, New York, 1977.
3. K. Hoffman and R. Kunze, *Linear algebra*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
4. G. Baumslag, *Wreath products and extensions*, Math. Z. **81** (1963), 286–299.

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER-SHEVA, ISRAEL (Current address)