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SOME FUNCTIONS WITH A UNIQUE INVARIANT MEAN

MICHEL TALAGRAND[1]

ABSTRACT. In a large class of groups, we construct a function which has a unique invariant mean, but which is not Riemann-measurable.

1. Introduction and notations. Let G be a compact group. For $f \in L^\infty = L^\infty(G)$ and $a \in G$, let $f_a \in L^\infty$ be given by $f_a(x) = f(ax)$ for $x \in G$. An *invariant mean* on L^∞ is a positive linear functional m such that $m(1) = 1$ and $m(f_a) = m(f)$ for $f \in L^\infty$, $a \in G$. The normalized Haar measure λ defines an invariant mean. We say $f \in L^\infty$ has a unique invariant mean if, for each invariant mean m, $m(f) = \int f\,d\lambda$. A sufficient condition for f to have a unique invariant mean is the following:

(1) For all $\varepsilon > 0$, there exist $a_1, \ldots, a_n$ in G with $\|(1/n)\Sigma_{i=1}^n f_{a_i} - \int f\,d\lambda\| < \varepsilon$.

If G is amenable as discrete, for example Abelian, it is known that this condition is also necessary.

It is clear that each continuous function has a unique invariant mean. An $f \in L^\infty$ will be said to be Riemann-measurable if it is (the class of) a function whose set of points of discontinuity is negligible. This is the same as saying that for all $\varepsilon > 0$, there exist continuous functions $g_1 < f < g_2$ with $\int(g_2 - g_1)\,d\lambda < \varepsilon$. Then f has a unique invariant mean. In fact, for each invariant mean m, we have $\int g_1\,d\lambda < m(f) < \int g_2\,d\lambda$ and both $\int g_1\,d\lambda$ and $\int g_2\,d\lambda$ differ from $\int f\,d\lambda$ by less than ε.

It is not hard to use (1) to construct in many groups functions f which are not Riemann-measurable and have a unique invariant mean. A natural question, raised by L. A. Rubel and A. L. Shields [1, p. 39], is: Suppose for each continuous g, the function fg has a unique invariant mean. Does it follow that f is Riemann-measurable? We are going to give a negative answer to this question in a large class of groups.

2. Construction. The Haar measure of a measurable set $A \subset G$ will be denoted by $|A|$. If $A, B \subset G$, let $AB = \{ab\colon a \in A, b \in B\}$.

THEOREM. *Suppose G satisfies the following condition*:

(2) *G is metrizable and there exist a sequence (S_n) of measurable sets whose diameters go to zero and a sequence of finite sets (P_n) of G such that $S_1 = G$, and*

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such that, for $n \geq 1$, the sets aS_{n+1} for $a \in P_{n+1}$ form a partition of S_n, and such that $Q_n = P_2 \cdots P_n$ is a finite subgroup of G.

Then there exists a function f: $G \to \{0, 1\}$, which is not Riemann-measurable, but such that fh has a unique invariant mean whenever h is Riemann-measurable.

In fact, by an obvious argument, it would be enough to suppose that G has a quotient which satisfies (2). It is also not hard to see that (2) is satisfied whenever $G = \mathbf{R}/\mathbf{Z}$ or $G = (\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$.

PROOF. Let $Q_n = P_2P_3 \cdots P_n$. Hence the sets aS_n for $a \in Q_n$ form a partition of G.

We shall construct by induction a sequence X_k of subsets of G, and an increasing sequence of integers n_k such that the following conditions hold:

(A_k) There exists $A \subset Q_{n_k}$ such that $X_k = AS_{n_k}$.

(B_k) For each $a \in Q_k$, $|X_k \cap aS_k| \geq |S_{n_k}|$, $|(G \setminus X_k) \cap aS_k| \geq |S_{n_k}|$.

(C_k) $|X_k \Delta X_{k-1}| \leq 2^{-k-2}|S_{n_{k-1}}|$ if $k \geq 2$.

(D_k) $\forall r \leq k \forall a \in Q_r$, $\sum_{b \in Q_{n_r}} \chi_{X_k \cap aS_r}(bx) =$ constant.

Before we proceed to the construction, let us show why it implies the result. From the conditions (C_k), it follows that there exists a measurable set $X \subset G$ with

$$|X \Delta X_k| \leq \sum_{l \geq k} |X_l \Delta X_{l+1}| \leq |S_{n_k}| \sum_{l \geq k} 2^{-l-3} \leq \tfrac{1}{2}|S_{n_k}|.$$

The conditions (B_k) show that for $a \in Q_k$, $|X \cap aS_k| > 0$ and $|(G \setminus X) \cap aS_k| > 0$. Since the diameter of S_k goes to zero, each open set contains a set aS_k for k large and $a \in Q_k$. Hence X and its complement meet each open set in a set of positive measure. It shows that $f = \chi_X$ is not Riemann-measurable.

Now if in condition (D_k) we fix r and let k go to infinity, we get that for all $a \in Q_r$, we have

$$\sum_{b \in Q_{n_r}} \chi_{aS_r}(bx) f(bx) = \text{constant}$$

and hence from (1) the function $\chi_{aS_r} f$ has a unique invariant mean. Now, each continuous function g can be uniformly approximated by a function of the type $\sum_{a \in Q_p} \alpha_a \chi_{aS_p}$ for p large; hence fg has a unique invariant mean. Now, if h is Riemann-measurable, for each $\varepsilon > 0$ there exist continuous functions g_1, g_2 with $g_1 \leq h \leq g_2$, $\int (g_2 - g_1)\, d\lambda \leq \varepsilon$, and hence, for each invariant mean m, $\int fg_1\, d\lambda \leq m(fh) \leq \int fg_2\, d\lambda$ where $\int f(g_2 - g_1)\, d\lambda \leq \varepsilon$, which shows that fh has a unique invariant mean.

We start the construction by picking $a \in P_2$, and setting $n_1 = 2$, $X_1 = aS_{n_1}$. Then (A_1), (B_1), (D_1) are satisfied and (C_1) is empty.

Now suppose the construction of X_k has been done, such that the conditions (A_k) to (D_k) are satisfied. Enumerate Q_{k+1} as $(a_q)_{1 \leq q \leq l}$ where $l = \operatorname{card} Q_{k+1}$. The construction of X_{k+1} will be done in $2l$ steps. We are going to construct for $p \leq 2l$ sets Y_p of G and increasing integers m_p, with $Y_1 = X_k$, $m_1 = n_k$, such that the following conditions hold.

(E_p) There exist $B \subset Q_{m_p}$ such that $Y_p = BS_{m_p}$.

(F_p) If $p = 2q$, then $|Y_p \cap a_q S_{k+1}| \geqslant |S_{m_p}|$. If $p = 2q - 1$ then $|(G \setminus Y_p) \cap a_q S_{k+1}| \geqslant |S_{m_p}|$.

(G_p) $|Y_p \Delta Y_{p-1}| \leqslant 2^{-p-2-k}|S_{m_{p-1}}|$ if $p \geqslant 2$.

(H_p) $\forall r \leqslant k$, $\forall a \in Q_r$, $\sum_{b \in Q_{n_r}} \chi_{Y_p \cap aS_r}(bx) =$ constant.

Before we perform the construction, let us show that $X_{k+1} = Y_{2l}$ satisfies (A_{k+1}) to (D_{k+1}) if we take $n_{k+1} = m_{2l}$. First, (A_{k+1}) is nothing else than (E_{2l}). Second, (D_{k+1}) is the union of (H_{2l}) and the following (which is clearly true)

$$\forall a \in Q_{k+1}, \sum_{b \in Q_{n_{k+1}}} \chi_{Y_{2l} \cap aS_{k+1}}(bx) = \text{constant}.$$

From the conditions (G_p) we get that

$$|X_k \Delta X_{k+1}| = |Y_1 \Delta Y_{2l}| \leqslant \sum_{p=2}^{2l} |Y_p \Delta Y_{p-1}| \leqslant \sum_{r \geqslant 2} 2^{-r-2-k}|S_{m_1}| \leqslant 2^{-k-3}|S_{n_k}|$$

and this is (C_{k+1}). By the same argument we have $|Y_p \Delta X_{k+1}| \leqslant \frac{1}{2}|S_{m_p}|$ for all p. Now if $a \in Q_{k+1}$, let q such that $a = a_q$. From (F_{2q}) we get

$$|X_{k+1} \cap aS_{k+1}| \geqslant |Y_{2q} \cap a_q S_{k+1}| - |Y_{2q} \Delta X_{k+1}| \geqslant \tfrac{1}{2}|S_{m_{2q}}|$$

and hence from (A_{k+1}), $|X_{k+1} \cap aS_{k+1}| \geqslant |S_{n_{k+1}}|$. We get similarly $|(G \setminus X_{k+1}) \cap aS_{k+1}| \geqslant |S_{n_{k+1}}|$ by considering $2q - 1$ instead of $2q$. Suppose now that (for example, the case $2q - 1$ being similar) Y_{2q} and m_{2q} have been constructed satisfying (E_{2q}) to (H_{2q}). If $|(G \setminus Y_{2q}) \cap a_{q+1}S_{k+1}| > 0$ then this measure is greater than $|S_{m_{2q}}|$ and it is enough to take $Y_{2q+1} = Y_{2q}$, $m_{2q+1} = m_{2q}$. If not, let m_{2q+1} be an integer such that

(3) 2^{2q+4+k} card $Q_k|S_{m_{2q+1}}| \leqslant |S_{m_{2q}}|$.

Since $|Y_1 \Delta Y_{2q}| \leqslant \frac{1}{2}|S_{m_1}|$ it follows from (B_k) that $|(G \setminus Y_{2q}) \cap vS_k| > 0$ for $v \in Q_k$. Take for v the unique element in Q_k such that $vS_k \supset a_{q+1}S_{k+1}$, i.e. such that $a_{q+1} = vw$ for a certain $w \in P_{k+1}$.

Set $Q' = P_{k+2}P_{k+3} \cdots P_{m_{2q}}$, $Q'' = P_{m_{2q}+1}P_{m_{2q}+2} \cdots P_{m_{2q+1}}$ and $S = S_{m_{2q}}$, $S' = S_{m_{2q+1}}$. Since $|(G \setminus Y_{2q}) \cap vS_k| > 0$, it follows from (E_{2q}) that there exist $w' \in P_{k+1}$ and $x \in Q'$ with $vw'xS \cap Y_{2q} = \varnothing$. Since we have assumed $a_{q+1}S_{k+1} \subset Y_{2q}$, we have $vwxS \subset Y_{2q}$. Now let y be any fixed element of Q''. We have $vw'xyS' \cap Y_{2q} = \varnothing$, $vwxyS' \subset Y_{2q}$. Each element $u \in Q = Q_{m_{2q+1}}$ can be written in a unique way, $u_1u_2u_3u_4$, where $u_1 \in Q_k$, $u_2 \in P_{k+1}$, $u_3 \in Q'$, $\mu_4 \in Q''$. Let T be the bijection of Q which lets u be fixed if $u_2 \notin \{w, w'\}$ or $u_3 \neq x$ or $u_4 \neq y$ and which exchanges $u_1w'xy$ and u_1wxy for each u_1.

From (E_{2q}) we have $Y_{2q} = BS$ for $B \subset Q_{m_{2q}}$ and hence $Y_{2q} = CS'$ where $C = BQ''$. Define $Y_{2q+1} = T(C)S'$. It is obvious that (E_{2q+1}) is satisfied. Since T moves at most 2cardQ_k points of Q, we have

$$|Y_{2q} \Delta Y_{2q+1}| \leqslant 2 \operatorname{card} Q_k |S'| \leqslant 2^{-2q-3-k}|S|$$

from condition (3); that is, (G_{2q+1}) holds and (F_{2q+1}) follows from the fact that, since $vw'xyS' \cap Y_{2q} = \varnothing$, we have $vwxyS' \cap Y_{2q+1} = \varnothing$ and $vw = a_{q+1}$.

To complete the proof, it remains to show that (H_{2q+1}) holds. Fix $r < k$ and $a \in Q_r$, and define

$$\alpha(x) = \sum_{b \in Q_{n_r}} \chi_{Y_{2q+1} \cap aS_r}(bx) = \sum_{\substack{b \in Q_{n_r} \\ t \in T(C)}} \chi_{tS' \cap aS_r}(bx).$$

We have $S_r = RS'$, where $R = P_{r+1}P_{r+2} \cdots P_{m_{2q+1}}$, so

$$\alpha(x) = \sum \chi_{tS' \cap at'S'}(bx),$$

where the summation is taken over $b \in Q_{n_r}$, $t \in T(C)$, $t' \in R$. We can write in a unique way $x = b'zx'$, where $x' \in S'$, $b' \in Q_{n_r}$, $z \in R' = P_{n_r+1}P_{n_r+2} \cdots P_{m_{2q+1}}$. Since Q_{n_r} is a group, we have

$$\alpha(x) = \sum \chi_{tS' \cap at'S'}(bzx')$$

where the summation is taken over $b \in Q_{n_r}$, $t \in T(C)$, $t' \in R$. We have

$$\chi_{tS' \cap at'S'}(bzx') = 1 \Leftrightarrow t = at' = bz.$$

Since T is a bijection, and $T^{-1} = T$, we have

$$\chi_{tS' \cap at'S'}(bzx') = \chi_{T(t)S' \cap T(at')S}(T(bz)x').$$

It is clear that $T(at') = aT'(t')$ where T' is a bijection of R. It is also easy to see that $T(bz) = T''(b)z'$ where T'' is a bijection of Q_{n_r} and $z' \in R'$ is independent of b. So we have

$$\alpha(x) = \sum \chi_{tS' \cap at'S'}(bz'x')$$

where the summation is taken over $t \in C$, $t' \in R$, $b \in Q_{n_r}$ so

$$\alpha(x) = \sum_{\substack{b \in Q_{n_r} \\ t \in C}} \chi_{tS' \cap aS_r}(bz'x') = \sum_{b \in Q_{n_r}} \chi_{Y_{2q} \cap aS_r}(bz'x').$$

From (H_{2q}) it follows that this quantity does not depend on $z'x'$, hence of x, which proves (H_{2q+1}) and concludes the proof.

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EQUIPE D'ANALYSE-TOUR 46, UNIVERSITÉ PARIS VI, 4 PLACE JUSSIEU, 75230 PARIS CEDEX 05, FRANCE