A NOTE ON THE THOM ISOMORPHISM

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ABSTRACT. The generalized homology version of the Thom isomorphism theorem is exploited to give easy proofs of several recent theorems.

Let $\Omega$ be an $H$-space with homotopy inverse $x \rightarrow x^{-1}$, and let $f : \Omega \rightarrow BF$ classify a stable spherical fibration over $\Omega$. The following result is proved in [2].

**Theorem 1.** If $f$ is an $H$-map, then the Thom spectrum $T(f)$ is a ring spectrum, and there is a homotopy equivalence $\alpha : T(f) \land \Omega_+ \rightarrow T(f) \land T(f)$ (where $+$ signifies a disjoint base point).

For many applications, it is importrant that $\alpha$ is given by the Thomification of an explicit map $g : \Omega \times \Omega \rightarrow \Omega \times \Omega$, where $g(x, y) = (xy^{-1}, y)$. For details, see [2].

Our purpose here is to note that Theorem 1 follows from a strong form of the Thom isomorphism theorem in generalized homology. Specifically, we assume given a stable spherical fibration $\nu : X \rightarrow BF$ over a locally finite CW complex $X$, and a ring spectrum $E$ orienting $\nu$; i.e. a Thom class $U : T(\nu) \rightarrow E$ whose restriction to a fibre $S^0 \hookrightarrow T(\nu)$ is the unit of $E$. Then we have

**Theorem 2.** There is a homotopy equivalence $\alpha(U) : E \land T(\nu) \rightarrow E \land X_+$ which on homotopy groups induces the traditional Thom isomorphism

$$\phi_{\nu} : E_{\ast}(T(\nu)) = \pi_{\ast}(E \land T(\nu)) \xrightarrow{\nu} \pi_{\ast}(E \land X_+) = E_{\ast}(X_+).$$

**Proof.** First suppose that $X$ has finite dimension, so that $\nu$ lifts to $\nu_n : X \rightarrow BF_n$ for suitably large $n$. Let $p_n : S(\nu_n) \rightarrow X$ be the associated $n$-sphere fibration, so that $T(\nu_n) = X \cup_{p_n} CS(\nu_n)$.

Now write $\Delta : T(\nu_n) \rightarrow T(\nu_n) \land X_+$ for the diagonal

$$x \mapsto \begin{cases} (x, p_n(x)), & x \neq \infty, \\ \infty, & x \neq \infty, \end{cases}$$

and consider the composite

$$\alpha(U) : E \land T(\nu_n) \xrightarrow{id \land \Delta} E \land T(\nu_n) \land X_+ \xrightarrow{id \land \Sigma^{n+1}E \land X_+ \mu \land id} E \land \Sigma^{n+1}X_+$$

wher $\mu$ is the product in $E$. 

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On homotopy groups this map induces a homomorphism $\phi_U: E_{*+n+1}(T(v_n)) \to E_*(X_+)$ which, by very construction, is $\cap U$. Thus $\phi_U$ is the usual Thom isomorphism, whence $\alpha(U)$ is an equivalence.

To complete the proof for all $X$, let $n \to \infty$.

**Corollary 3.** Theorem 1 is true.

**Proof.** Let $X = \Omega$; choose $E = T(v)$ and $U$ as the identity. Then $\alpha(id): T(v) \cap T(v) \to T(v) \cap \Omega_+$ is just the Thomification of $g': \Omega \times \Omega \to \Omega \times \Omega$ given by $g'(x, y) = (xy, y)$. Thus the inverse of $\alpha(id)$ is the Thomification of $g^{-1}$, and $g^{-1}$ is clearly $g$.

**Corollary 4.** Suppose $v: X \to BO$ lifts to the $(i - 1)$-connected cover $BO(i)$. Let $MO(i)$ be the corresponding Thom spectrum. Then there is an equivalence $\beta: MO(i) \wedge T(v) \to MO(i) \wedge X_+$.

Applying this, we deduce a result of [1].

**Corollary 5.** Let $N(i)$ be such that $N(i)\xi$ is trivial over $RP^1$, where $\xi$ is the Hopf line bundle. Then there is an equivalence $MO(i) \wedge RP^\infty_{N(i)+j} \to MO(i) \wedge \Sigma^{N(i)}RP^\infty_j$ for all $j$ (where $RP^m_n = RP^m / RP^{n-1}$).

**Proof.** Consider the composite

$$
MO(i) \wedge RP^\infty_{N(i)+j} \to MO(i) \wedge RP^\infty_{N(i)} \wedge RP^\infty_j
$$

$$
\cong \to \beta \wedge \text{id}

\to \to \cong

MO(i) \wedge \Sigma^{N(i)}RP^\infty_+ \wedge RP^\infty_j \to MO(i) \wedge \Sigma^{N(i)}RP^\infty_j,
$$

where the first map is induced by the diagonal $T(N(i)\xi \oplus j\xi) \to T(N(i)\xi) \wedge T(j\xi)$, and the third by the projection $RP^\infty_+ \to S^0$.

A simple homology calculation shows the composite to be an equivalence.

**References**


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