FROM A 3-LOCAL PLUS 3-FUSION
TO THE CENTRALIZER OF AN INVOLUTION

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Abstract. It is shown that much of the structure of the centralizer of a central
involution in a group of characteristic 2 type with a standard 3-component of type
$GL(n, 2)$, $n > 6$, is easily determined from the 3-fusion. Consequently, one can
shorten the previous treatment of such groups by Finkelstein and the author.

In [4], Gorenstein and Lyons reduced part of the classification problem for finite
simple groups of characteristic 2 type to the solution of certain standard form
problems for odd primes. One such problem was to find simple groups with a
standard 3-component of type $GL(n, 2)$, $n > 6$, subject to a few side conditions.
This problem was solved by Finkelstein and Frohardt in [1]. The purpose of this
paper is to simplify the treatment given there. In particular, the last four sections of
[1] can be replaced by the argument here by using results of Timmesfeld [7] and
Smith [5] on groups with large extraspecial subgroups. These results are used
elsewhere in the developing characterization of finite simple groups of characteristic
2 type, so the present paper might be incorporated into a shorter overall
treatment of such groups. Gorenstein and Lyons use a similar approach in §8 of [4]
to eliminate certain sporadic standard form problems, but their argument is
different from ours.

To state the main result of this paper, recall the notation of [1]. Let $L \cong
GL(n, 2)$, $n > 6$, and let $\langle b_1, \ldots, b_r \rangle$, $r = \lceil n/2 \rceil$, be an elementary abelian 3-sub-
group of $L$ or $r$ such that each $b_i$ belongs to a natural $GL(2, 2)$ subgroup of $L$.
The main theorem here is the following.

Theorem A. Let $G$ be a finite simple group of characteristic 2 type that contains $L$,
$L \cong GL(n, 2)$, and an element $b$ of order 3 with $b \in C(L)$. Assume that (a), (b), and
(c) hold.

(a) $L \leq C(b)$.
(b) $C(L)$ has cyclic Sylow 3-subgroups.
(c) $\text{Aut}_G(\langle b, b_1, \ldots, b_r \rangle)$ contains the monomial group on $\{b, b_1, \ldots, b_r\}$.

Then $C(L)$ contains an involution $t$ such that $O_2(C(t))$ is extraspecial. Furthermore,
either $|O_2(C(t))| = 2^{2n+1}$ or $n = 6$ and $|O_2(C(t))| = 2^{21}$.

Propositions 1 and 2 of [1] show that assumption (c) follows from (a), (b), and
the following assumptions.
(d) $\langle b \rangle$ is not strongly closed in $C(b)$; and
(e) $m_{2,3}(G) = r + 1$.

Here $m_{2,3}(G)$ denotes the largest rank of an elementary abelian 3-subgroup of a 2-local subgroup of $G$.

Using Propositions 1 and 2 of [1], and results of Timmesfeld [7] and Smith [5], we have the following corollary to the theorem.

**Corollary.** Assume that $G$ is a finite simple group of characteristic 2 type which contains a subgroup $L \cong \text{GL}(n, 2)$, $n > 6$, and an element $b$ of order 3 satisfying conditions (a), (b), (d), and (e) above. Then either $G \cong \text{GL}(n + 2, 2)$, or $n = 6$ and $G \cong E_6(2)$.

This corollary is the main result of [1].

Without using Timmesfeld’s theorem, it is still possible to characterize $G$ fairly quickly in the case that $|O_2(C(t))| = 2^{2n+1}$. A sketch of such an argument, using Suzuki’s characterization of $\text{GL}(n + 2, 2)$ [6], is given at the end of this paper.

The argument here is based on the fact that $\text{GL}(3, 2)$ cannot act on a nontrivial 2-group with the elements of order 3 acting fixed point free. This, together with the assumption that $G$ has characteristic 2 type, is used to control $O_2(C(t))$ for an appropriate involution $t$. We shall use without explicit reference the assumption that $N(\langle b, b_1, b_2, \ldots, b_r \rangle)$ is $r + 1$-transitive on $\{b, b_1, \ldots, b_r\}$. We shall also use the following properties of $L$ which can be verified by straightforward computation.

(L1) If $t_1$ is an involution in $L$ that centralizes $\langle b_2, b_3, \ldots, b_r \rangle$ and inverts $b_1$, then $C_L(t_1) = T_1L_1$ where $T_1 = O_2(C_L(t_1))$ is extraspecial of type $2^{2(n-2)+1}$ and $L_1 = E(C_L(b_1)) \equiv C_L(b_1)/\langle b_1 \rangle \cong \text{GL}(n - 2, 2)$. $[T_1, b_2]$ is extraspecial of type $2^+$ and is centralized by $b_3$. For each $i > 2$, we can find $K = \text{GL}(3, 2)$ with $b_i \subseteq K < L_1$.

(L2) $\text{Aut } L = \text{Inn } L(b)$ where $b$ is an involution that centralizes $\langle b_1, \ldots, b_r \rangle$ but does not centralize $L_1$. We have $C_L(b_i) \cong \text{Sp}(2r, 2)$.

**Lemma 1.** $O_3(C(b))$ has odd order.

**Proof.** Let $X \in \text{Syl}_2(O_3(C(b)))$. Then $X$ centralizes $L$. Either $X \in \text{Syl}_2(C(B))$ or $C(B)$ contains an element $\sigma \in N(X)$ such that $\langle \sigma^2 \rangle = C_{\langle \sigma \rangle}(L) < X$ and $\langle L, \sigma \rangle/\langle \sigma^2 \rangle \cong \text{Aut } \text{GL}(n, 2)$. In any case $X \in \text{Syl}_2(O_3(C(b, b_1)))$, so $X \in \text{Syl}_2(O_3(\langle b, b_1 \rangle))$. Set $Y = O_2(N(X))$. Then $L$ acts on $Y$ and $C_Y(b_1) < O_2(C(b_1) \cap N(X)) = X$.

Therefore $Y = X$ because $b_1$ belongs to a $\text{GL}(3, 2)$ subgroup of $L$. This implies that $X = 1$ because 2-local subgroups of $G$ are 2-constrained.

**Lemma 2.** $N(\langle b \rangle)$ contains an involution $t$ such that $t \in C(L)$ and $t \in E(O_3^+(C(b_i)))$, $i = 1, \ldots, r$.

**Proof.** Let $L_i = E(O_3^+(C(b_i)))$. Then $L_i$ contains an involution $t$ such that $t$ inverts $b$ and centralizes $\langle b_2, \ldots, b_r \rangle$ and $C_{L_i}(t) = T_1 \cdot L^*$ where $T_1$ is extraspecial.
of type $2^{2(n-2)+1}$ and $L^* \cong GL(n-2, 2)$. We have

$$C_L(\langle t, b_i \rangle) \cong 2^{2(n-4)+1} GL(n-4, 2)$$

for $i = 2, \ldots, r$ so that $t \in O^3(C(\langle b_1, b_i \rangle))$, $i = 2, \ldots, r$. Therefore $t \in O^3(C(b_i))$, which implies that $t \in E(O^3(C(b_i)))$. It remains to show that $t$ centralizes $L$. Certainly $t$ acts on $L$ and centralizes $\langle b_1, \ldots, b_r \rangle$. But $L^* \leq C_L(t)$ and $L^* \cong GL(n-2, 2)$. It follows from properties of $Aut(\text{GL}(n, 2))$ that $t$ centralizes $L$.

**Lemma 3.** If $n > 7$ and $L = GL(n, 2)$ acts on a 2-group $T$ so that $[C_T(b_1), b_2] < C_T(b_3)$, then $T = C_T(b_2) \cdot C_T(b_3)$. (As above, $\langle b_1 \rangle$, $\langle b_2 \rangle$, and $\langle b_3 \rangle$ are distinct, commuting subgroups of order 3 contained in natural $GL(2, 2)$ subgroups.)

**Proof.** By induction on $|T|$ and coprime action, we may assume that $T$ is abelian. Set $T^* = \langle T, b_2, b_3 \rangle$ and $L^* = E(C_L(\langle b_2, b_3 \rangle))$. Then $L^* \cong GL(n-4, 2)$ acts on $T^*$ and $C_{T^*}(b_1) < [C_T(b_1), b_2, b_3] = 1$. Since $b_1$ belongs to a $GL(3, 2)$ subgroup of $L^*$, this implies that $T^* = 1$. Therefore $[T, b_2] < C_T(b_3)$ and $T = C_T(b_2)C_T(b_3)$.

We shall use the following notation for the remainder of this paper. Let $T = O_2(C(t))$ where $t$ is an involution that inverts $b$, centralizes $L$, and belongs to $E(O^3(C(b_i)))$, as in Lemma 2. Let $T_i = C_T(b_i)$ and $S_i = O_2(C(t) \cap C(b_i))$, $i = 1, \ldots, r$. Note that $T_i < S_i$ for all $i$. By properties of $L$, we have that $S_i$ is extraspecial of type $2^{2(n-2)+1}$ and that $[S_1, b_2]$ is extraspecial of width 2. Also, $S_1 \cap S_2$ is extraspecial of width $n-4$. Finally, $[S_1, b_2] < S_3$ so that $S_1 = (S_1 \cap S_2) \cdot (S_1 \cap S_3)$.

**Lemma 4.** If $n > 7$, then $T$ is extraspecial of type $2^{2n+1}$.

**Proof.** Using the notation just introduced, we have $[C_T(b_1), b_2] < [S_1, b_2] < C(b_2)$. Therefore $T = T_2T_3$ by Lemma 3. In fact, $T = T_iT_j$ for $i \neq j$, $1 < i, j < r$.

Set $\overline{C(i)} = C(i)/\langle t \rangle$. Then $\overline{T} \leq \overline{S_iS_j}$ for $i \neq j$. Since $\overline{S_i}$ is abelian, we have $\overline{S_i} \cap \overline{S_j} < \overline{C_{C_0}(\overline{C(i)})}$. But $\overline{C_{C_0}(\overline{T})} < \overline{O_2(\overline{C(i)})} = \overline{T}$ because $T = O_2(C(t))$ and $G$ has characteristic 2 type. Therefore $S_1 = \langle S_1 \cap S_i \mid i \neq 1 \rangle < T$. This implies that $T = S_1S_2$. Furthermore, $\overline{S_1} = \langle \overline{S_1} \cap \overline{S_2} \mid i \neq 1 \rangle < Z(\overline{T})$, so $\overline{T}$ is elementary abelian. This implies that $\langle t \rangle = T' = \Phi(T)$ since $S_1$ is evidently nonabelian. Because $C_{Z(T)}(b_1) < Z(T) = \langle t \rangle$, and $b_1$ belongs to a $GL(3, 2)$ subgroup of $L$, it follows that $Z(T) = \langle t \rangle$ and $T$ is extraspecial. Since $S_i$ is extraspecial of type $2^{2(n-2)+1}$, $i = 1, 2$, and $S_1 \cap S_2$ is extraspecial of width $n-4$, it follows from elementary considerations that $T = S_1S_2$ is extraspecial of order $2^{2n+1}$.

**Lemma 5.** If $n = 6$, then one of the following holds.

(a) $T$ is extraspecial of type $2^{13}$.

(b) $T$ is extraspecial of type $2^{21}$ and $L$ acts irreducibly on $T/\langle t \rangle$.

**Proof.** As before, we have $T_1 = (T_1 \cap S_2) \cdot (T_1 \cap S_3)$. Since $S_1 \cap S_2 \cong O_{2+} \cdot O_{2+}$, we have that $|T_1 \cap S_2| = |T_1 \cap T_2| = 2^{2a+1}$ for $a = 0, 1, 2$. Setting $\overline{C(i)} = C(i)/\langle t \rangle$, this implies that $|\overline{T_1}| = 2^{2a}$.
Let \( H = N_L(\langle b_1, b_2, b_3 \rangle) \). Then \( H \cong \Sigma_3 \times \Sigma_3 \) and \( H \) acts on \( \overline{T} \). We have \( C_T(\langle b_1, b_2 \rangle) = E_{2a} \) and \( C_T(\langle b_1, b_2, b_3 \rangle) \subseteq O_2(C(t) \cap C(\langle b_1, b_2, b_3 \rangle)) = \langle t \rangle \). Thus \( C_T(\langle b_1, b_2, b_3 \rangle) = 1 \) so that \( H \) acts faithfully on all \( H \)-factors of \( \overline{T} \). It follows from (3.5) of [3] that there are \( \alpha H \)-factors of \( \overline{T} \) of dimension 6 on each of which \( b_1 \) fixes an \( E_{16} \)-subgroup and that the remaining \( H \)-factors have dimension 8. Assume that there are \( \beta H \)-factors of dimension 8. Then \( |\overline{T}| = 2^{2a+8}\beta \).

Now let \( K \) be a \( GL(3, 2) \)-subgroup of \( C_L(b_2) \) with \( b_1 \in K \). Then \( K \) acts on \( \overline{T}_2 \cong E_{2a} \) and \( |C_{\overline{T}}(b_1)| = 2^{2a} \). Let \( \{R_k\} \) be the \( K \)-factors of \( \overline{T} \) not involved in \( \overline{T}_2 \) for some chief \( K \)-series of \( \overline{T} \). Then \( \Sigma \dim R_k = 2\alpha + 8\beta \), and \( \Sigma \dim C_{R_k}(b_1) = 2\alpha \).

By inspecting the list of irreducible GF(2) \( K \)-modules, we have \( \dim R_k < 4 \dim C_{R_k}(b_1) \) for all \( k \). Thus \( 2\alpha + 8\beta < 8\alpha \), so \( \beta < \alpha \).

If \( \beta = 0 \), then \( T = T_2, T_3 \) and the argument of the previous lemma shows that \( T \) is extraspecial of type \( 2^4_+ \). We may therefore assume that \( \beta = 1 \), whence \( \alpha = 2 \) and \( |T| = 2^{21} \).

We need to show that \( L \) acts irreducibly on \( \overline{T} \). The factors of a chief \( H \)-series for \( \overline{T} \) are of dimensions 6, 6, and 8, and \( b_1 \) has no fixed points on the 8-dimensional section. Therefore the 8-dimensional section does not admit \( L \), so either \( L \) acts irreducibly on \( \overline{T} \), or one of the 6-dimensional \( H \)-sections \( T^* \) admits \( L \). Since \( |C_{T^*}(b_1)| = 16 \), \( T^* \) must involve 3 trivial \( K \)-sections and one 3-dimensional \( K \)-section, where \( b_1 \in K \cong GL(3, 2) \), as above. But \( K \) is contained in a \( GL(4, 2) \) subgroup \( L_2 \) of \( C(b_2) \) which acts on \( \overline{T}_2 \cong 2^9_+ \). We have that \( \overline{T}_2 \) is the sum of two 4-dimensional GF(2) \( L_2 \)-modules. Each of these \( L_2 \)-modules is the sum of a trivial \( K \)-module and a 3-dimensional \( K \)-module. At least one of these \( L_2 \)-modules, say \( S^* \), is not involved in \( T^* \). Between \( S^* \) and \( T^* \), we now have 4 trivial \( K \)-sections of \( \overline{T} \) and 2 3-dimensional \( K \)-sections of \( \overline{T} \) all of which occur in a chief \( K \)-series for \( \overline{T} \).

Letting \( \{S_k\} \) be the remaining \( K \)-sections of \( \overline{T} \) in this chief \( K \)-series, we have \( \Sigma \dim S_k = 10 \) and \( \Sigma \dim C_{S_k}(b_1) = 2 \). But \( \dim S_k < 4 \dim C_{S_k}(b_1) \) because \( S_k \) is an irreducible GF(2) \( K \)-module. This is a contradiction, so we have that \( L \) acts irreducibly on \( \overline{T} \).

Since \( T \) is evidently nonabelian, \( T \) must be extraspecial. As \( C_T(b_1) = 2^9_+ \), we have that \( T \) has type \( + \).

Theorem A is an immediate consequence of Lemmas 4 and 5.

To prove the Corollary, we use a deep result of Timmesfeld [7] which reduces the general classification of simple groups possessing an involution \( z \) with \( O_2(C(z)) = F^*(C(z)) \) extraspecial to several more specific configurations. By inspecting the list and using the 3-local information at hand together with \( L < C(t) \), we have that \( G \cong GL(n + 2, 2) \) when \( |T'| = 2^{2n+1} \), while if \( n = 6 \) and \( |T'| = 2^{21} \), then \( C(t) = TL(\sigma) \) where \( \sigma^2 = 1 \). By Smith [5], we have \( G \cong E_6(2) \) in this latter case.

The case \( |T'| = 2^{2n+1} \) could also be handled as follows. By an argument similar to that in Lemma 4.5 and Proposition 4.6 of [2], we have \( C(t) = TL(\sigma) \) where \( \sigma^2 = 1 \) and \( TL \) is isomorphic to the centralizer of an involution in \( GL(n + 2, 2) \). Suppose \( C(t) \neq CT \). Then we can choose \( \sigma \) so that \( C_L(\sigma) = Sp(2n, 2) \). In particular, \( O_2(C(\sigma) \cap C(t)) = \langle \sigma \rangle C_T(\sigma) \). Using \( [T, b_1] \leq C_T(b_2) \), it is not difficult to show that \( C_T(\sigma) \cap O_2(C(\sigma)) = 1 \). The \( P \times Q \) lemma applied to \( \langle b_1 \rangle \times \langle t \rangle \) acting on
$O_2(C(\sigma))$ then implies that $b_1$ centralizes $O_2(C(\sigma))$, contradicting the assumption that $G$ has characteristic 2 type. Thus $C(\iota) = TL$ and $G \cong GL(n + 2, 2)$ by Suzuki [6].

REFERENCES


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