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FROM A 3-LOCAL PLUS 3-FUSION TO THE CENTRALIZER OF AN INVOLUTION

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ABSTRACT. It is shown that much of the structure of the centralizer of a central involution in a group of characteristic 2 type with a standard 3-component of type $GL(n, 2)$, $n > 6$, is easily determined from the 3-fusion. Consequently, one can shorten the previous treatment of such groups by Finkelstein and the author.

In [4], Gorenstein and Lyons reduced part of the classification problem for finite simple groups of characteristic 2 type to the solution of certain standard form problems for odd primes. One such problem was to find simple groups with a standard 3-component of type $GL(n, 2)$, $n > 6$, subject to a few side conditions. This problem was solved by Finkelstein and Frohardt in [1]. The purpose of this paper is to simplify the treatment given there. In particular, the last four sections of [1] can be replaced by the argument here by using results of Timmesfeld [7] and Smith [5] on groups with large extraspecial subgroups. These results are used elsewhere in the developing characterization of finite simple groups of characteristic 2 type, so the present paper might be incorporated into a shorter overall treatment of such groups. Gorenstein and Lyons use a similar approach in §8 of [4] to eliminate certain sporadic standard form problems, but their argument is different from ours.

To state the main result of this paper, recall the notation of [1]. Let $L \cong GL(n, 2)$, $n > 6$, and let $\langle b_1, \ldots, b_r \rangle$, $r = [n/2]$, be an elementary abelian 3-subgroup of L or r such that each b_i belongs to a natural $GL(2, 2)$ subgroup of L.

The main theorem here is the following.

THEOREM A. *Let G be a finite simple group of characteristic 2 type that contains L, $L \cong GL(n, 2)$, and an element b of order 3 with $b \in C(L)$. Assume that* (a), (b), *and* (c) *hold.*

(a) $L \trianglelefteq C(b)$.

(b) *$C(L)$ has cyclic Sylow 3-subgroups.*

(c) $\mathrm{Aut}_G(\langle b, b_1, \ldots, b_r \rangle)$ *contains the monomial group on* $\{b, b_1, \ldots, b_r\}$.

Then $C(L)$ contains an involution t such that $O_2(C(t))$ is extraspecial. Furthermore, either $|O_2(C(t))| = 2^{2n+1}$ or $n = 6$ and $|O_2(C(t))| = 2^{21}$.

Propositions 1 and 2 of [1] show that assumption (c) follows from (a), (b), and the following assumptions.

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(d) $\langle b \rangle$ *is not strongly closed in* $C(b)$; *and*
(e) $m_{2,3}(G) = r + 1$.

Here $m_{2,3}(G)$ denotes the largest rank of an elementary abelian 3-subgroup of a 2-local subgroup of G.

Using Propositions 1 and 2 of [1], and results of Timmesfeld [7] and Smith [5], we have the following corollary to the theorem.

COROLLARY. *Assume that G is a finite simple group of characteristic* 2 *type which contains a subgroup $L \cong GL(n, 2)$, $n \geqslant 6$, and an element b of order* 3 *satisfying conditions* (a), (b), (d), *and* (e) *above. Then either $G \cong GL(n + 2, 2)$, or $n = 6$ and $G \cong E_6(2)$.*

This corollary is the main result of [1].

Without using Timmesfeld's theorem, it is still possible to characterize G fairly quickly in the case that $|O_2(C(t))| = 2^{2n+1}$. A sketch of such an argument, using Suzuki's characterization of $GL(n + 2, 2)$ [6], is given at the end of this paper.

The argument here is based on the fact that $GL(3, 2)$ cannot act on a nontrivial 2-group with the elements of order 3 acting fixed point free. This, together with the assumption that G has characteristic 2 type, is used to control $O_2(C(t))$ for an appropriate involution t. We shall use without explicit reference the assumption that $N(\langle b, b_1, b_2, \ldots, b_r \rangle)$ is $r + 1$-transitive on $\{b, b_1, \ldots, b_r\}$. We shall also use the following properties of L which can be verified by straightforward computation.

(L1) If t_1 is an involution in L that centralizes $\langle b_2, b_3, \ldots, b_r \rangle$ and inverts b_1, then $C_L(t_1) = T_1 L_1$ where $T_1 = O_2(C_L(t_1))$ is extraspecial of type $2_+^{2(n-2)+1}$ and $L_1 = E(C_L(b_1)) \cong C_L(b_1)/\langle b_1 \rangle \cong GL(n - 2, 2)$. $[T_1, b_2]$ is extraspecial of type 2_+^5 and is centralized by b_3. For each $i \geqslant 2$, we can find $K \cong GL(3, 2)$ with $b_i \in K \leqslant L_1$.

(L2) Aut L = Inn $L\langle \sigma \rangle$ where σ is an involution that centralizes $\langle b_1, \ldots, b_r \rangle$ but does not centralize L_1. We have $C_L(\sigma) \cong \mathrm{Sp}(2r, 2)$.

LEMMA 1. *$O_{3'}(C(b))$ has odd order.*

PROOF. Let $X \in \mathrm{Syl}_2(O_{3'}(C(b)))$. Then X centralizes L. Either $X \in \mathrm{Syl}_2(C(B))$ or $C(B)$ contains an element $\sigma \in N(X)$ such that $\langle \sigma^2 \rangle = C_{\langle \sigma \rangle}(L) \leqslant X$ and $\langle L, \sigma \rangle / \langle \sigma^2 \rangle \cong \mathrm{Aut}\, GL(n, 2)$. In any case $X \in \mathrm{Syl}_2(O_{3'}(C(\langle b, b_1 \rangle)))$, so $X \in \mathrm{Syl}_2(O_{3'}(C(b_1)))$. Set $Y = O_2(N(X))$. Then L acts on Y and

$$C_Y(b_1) \leqslant O_2(C(b_1) \cap N(X)) = X.$$

Therefore $Y = X$ because b_1 belongs to a $GL(3, 2)$ subgroup of L. This implies that $X = 1$ because 2-local subgroups of G are 2-constrained.

LEMMA 2. *$N(\langle b \rangle)$ contains an involution t such that $t \in C(L)$ and $t \in E(O^{3'}(C(b_i)))$, $i = 1, \ldots, r$.*

PROOF. Let $L_1 = E(O^{3'}(C(b_1)))$. Then L_1 contains an involution t such that t inverts b and centralizes $\langle b_2, \ldots, b_r \rangle$ and $C_{L_1}(t) = T_1 \cdot L^*$ where T_1 is extraspecial

of type $2_+^{2(n-2)+1}$ and $L^* \cong GL(n-2, 2)$. We have

$$C_{L_1}(\langle t, b_i\rangle) \cong 2_+^{2(n-4)+1}\, GL(n-4, 2)$$

for $i = 2, \ldots, r$ so that $t \in O^{3'}(C(\langle b_1, b_i\rangle))$, $i = 2, \ldots, r$. Therefore $t \in O^{3'}(C(b_i))$, which implies that $t \in E(O^{3'}(C(b_i)))$. It remains to show that t centralizes L. Certainly t acts on L and centralizes $\langle b_1, \ldots, b_r\rangle$. But $L^* \lessdot C_L(t)$ and $L^* \cong GL(n-2, 2)$. It follows from properties of Aut $GL(n, 2)$ that t centralizes L.

LEMMA 3. *If $n \geqslant 7$ and $L \cong GL(n, 2)$ acts on a 2-group T so that $[C_T(b_1), b_2] \lessdot C_T(b_3)$, then $T = C_T(b_2) \cdot C_T(b_3)$. (As above, $\langle b_1\rangle, \langle b_2\rangle$, and $\langle b_3\rangle$ are distinct, commuting subgroups of order 3 contained in natural $GL(2, 2)$ subgroups.)*

PROOF. By induction on $|T|$ and coprime action, we may assume that T is abelian. Set $T^* = [T, b_2, b_3]$ and $L^* = E(C_L(\langle b_2, b_3\rangle))$. Then $L^* \cong GL(n-4, 2)$ acts on T^* and $C_{T^*}(b_1) \lessdot [C_T(b_1), b_2, b_3] = 1$. Since b_1 belongs to a $GL(3, 2)$ subgroup of L^*, this implies that $T^* = 1$. Therefore $[T, b_2] \lessdot C_T(b_3)$ and $T = C_T(b_2)C_T(b_3)$.

We shall use the following notation for the remainder of this paper. Let $T = O_2(C(t))$ where t is an involution that inverts b, centralizes L, and belongs to $E(O^{3'}(C(b_1)))$, as in Lemma 2. Let $T_i = C_T(b_i)$ and $S_i = O_2(C(t) \cap C(b_i))$, $i = 1, \ldots, r$. Note that $T_i \lessdot S_i$, for all i. By properties of L, we have that S_i is extraspecial of type $2_+^{2(n-2)+1}$ and that $[S_1, b_2]$ is extraspecial of width 2. Also, $S_1 \cap S_2$ is extraspecial of width $n-4$. Finally, $[S_1, b_2] \lessdot S_3$ so that $S_1 = (S_1 \cap S_2) \cdot (S_1 \cap S_3)$.

LEMMA 4. *If $n \geqslant 7$, then T is extraspecial of type 2_+^{2n+1}.*

PROOF. Using the notation just introduced, we have $[C_T(b_1), b_2] \lessdot [S_1, b_2] \lessdot C(b_3)$. Therefore $T = T_2T_3$ by Lemma 3. In fact, $T = T_iT_j$ for $i \neq j$, $1 \lessdot i, j \lessdot r$.

Set $\overline{C(t)} = C(t)/\langle t\rangle$. Then $\overline{T} \lessdot \overline{S}_i\overline{S}_j$ for $i \neq j$. Since $\overline{S}_i$ is abelian, we have $\overline{S}_i \cap \overline{S}_j \lessdot C_{\overline{C(t)}}(\overline{T})$. But $C_{\overline{C(t)}}(\overline{T}) \lessdot O_2(\overline{C(t)}) = \overline{T}$ because $T = O_2(C(t))$ and G has characteristic 2 type. Therefore $S_1 = \langle S_1 \cap S_i\colon i \neq 1\rangle \lessdot T$. This implies that $T = S_1S_2$. Furthermore, $\overline{S}_1 = \langle \overline{S}_1 \cap \overline{S}_i\colon i \neq 1\rangle \lessdot Z(\overline{T})$, so $\overline{T}$ is elementary abelian. This implies that $\langle t\rangle = T' = \Phi(T)$ since S_1 is evidently nonabelian. Because $C_{Z(T)}(b_1) \lessdot Z(T_1) = \langle t\rangle$, and b_1 belongs to a $GL(3, 2)$ subgroup of L, it follows that $Z(T) = \langle t\rangle$ and T is extraspecial. Since S_i is extraspecial of type $2_+^{2(n-2)+1}$, $i = 1, 2$, and $S_1 \cap S_2$ is extraspecial of width $n-4$, it follows from elementary considerations that $T = S_1S_2$ is extraspecial of order 2^{2n+1}.

LEMMA 5. *If $n = 6$, then one of the following holds.*

(a) *T is extraspecial of type 2_+^{13}.*

(b) *T is extraspecial of type 2_+^{21} and L acts irreducibly on $T/\langle t\rangle$.*

PROOF. As before, we have $T_1 = (T_1 \cap S_2) \cdot (T_1 \cap S_3)$. Since $S_1 \cap S_2 \cong Q_8 * Q_8$, we have that $|T_1 \cap S_2| = |T_1 \cap T_2| = 2^{2\alpha+1}$ for $\alpha = 0, 1$, or 2. Setting $\overline{C(t)} = C(t)/\langle t\rangle$, this implies that $|\overline{T}_i| = 2^{4\alpha}$.

Let $H = N_L(\langle b_1, b_2, b_3\rangle)$. Then $H \cong \Sigma_3 \wr \Sigma_3$ and H acts on $\bar{T}$. We have $C_{\bar{T}}(\langle b_1, b_2\rangle) \cong E_{2^{2\alpha}}$ and $C_T(\langle b_1, b_2b_3\rangle) \subseteq O_2(C(t) \cap C(\langle b_1, b_2b_3\rangle)) = \langle t\rangle$. Thus $C_{\bar{T}}(\langle b_1, b_2b_3\rangle) = 1$ so that H acts faithfully on all H-factors of $\bar{T}$. It follows from (3.5) of [3] that there are α H-factors of $\bar{T}$ of dimension 6 on each of which b_1 fixes an E_{16}-subgroup and that the remaining H-factors have dimension 8. Assume that there are β H-factors of dimension 8. Then $|\bar{T}| = 2^{6\alpha+8\beta}$.

Now let K be a $GL(3, 2)$-subgroup of $C_L(b_2)$ with $b_1 \in K$. Then K acts on $\bar{T}_2 \cong E_{2^{4\alpha}}$ and $|C_{\bar{T}_2}(b_1)| = 2^{2\alpha}$. Let $\{R_k\}$ be the K-factors of $\bar{T}$ not involved in $\bar{T}_2$ for some chief K-series of $\bar{T}$. Then $\Sigma \dim R_k = 2\alpha + 8\beta$, and $\Sigma \dim C_{R_k}(b_1) = 2\alpha$. By inspecting the list of irreducible $GF(2)$ K-modules, we have $\dim R_k \leqslant 4 \dim C_{R_k}(b_1)$ for all k. Thus $2\alpha + 8\beta \leqslant 8\alpha$, so $\beta < \alpha$.

If $\beta = 0$, then $T = T_2T_3$ and the argument of the previous lemma shows that T is extraspecial of type 2_+^{13}. We may therefore assume that $\beta = 1$, whence $\alpha = 2$ and $|T| = 2^{21}$.

We need to show that L acts irreducibly on $\bar{T}$. The factors of a chief H-series for $\bar{T}$ are of dimensions 6, 6, and 8, and b_1 has no fixed points on the 8-dimensional section. Therefore the 8-dimensional section does not admit L, so either L acts irreducibly on $\bar{T}$, or one of the 6-dimensional H-sections T^* admits L. Since $|C_{T^*}(b_1)| = 16$, T^* must involve 3 trivial K-sections and one 3-dimensional K-section, where $b_1 \in K \cong GL(3, 2)$, as above. But K is contained in a $GL(4, 2)$ subgroup L_2 of $C(b_2)$ which acts on $T_2 \cong 2_+^9$. We have that $\bar{T}_2$ is the sum of two 4-dimensional $GF(2)L_2$-modules. Each of these L_2-modules is the sum of a trivial K-module and a 3-dimensional K-module. At least one of these L_2-modules, say S^*, is not involved in T^*. Between S^* and T^*, we now have 4 trivial K-sections of $\bar{T}$ and 2 3-dimensional K-sections of $\bar{T}$ all of which occur in a chief K-series for $\bar{T}$. Letting $\{S_k\}$ be the remaining K-sections of $\bar{T}$ in this chief K-series, we have $\Sigma \dim S_k = 10$ and $\Sigma \dim C_{S_k}(b_1) = 2$. But $\dim S_k \leqslant 4 \dim C_{S_k}(b_1)$ because S_k is an irreducible $GF(2)$ K-module. This is a contradiction, so we have that L acts irreducibly on $\bar{T}$.

Since T is evidently nonabelian, T must be extraspecial. As $C_T(b_1) \cong 2_+^9$, we have that T has type $+$.

Theorem A is an immediate consequence of Lemmas 4 and 5.

To prove the Corollary, we use a deep result of Timmesfeld [7] which reduces the general classification of simple groups possessing an involution z with $O_2(C(z)) = F^*(C(z))$ extraspecial to several more specific configurations. By inspecting the list and using the 3-local information at hand together with $L \leqslant C(t)$, we have that $G \cong GL(n + 2, 2)$ when $|T| = 2^{2n+1}$, while if $n = 6$ and $|T| = 2^{21}$, then $C(t) = TL\langle\sigma\rangle$ where $\sigma^2 = 1$. By Smith [5], we have $G \cong E_6(2)$ in this latter case.

The case $|T| = 2^{2n+1}$ could also be handled as follows. By an argument similar to that in Lemma 4.5 and Proposition 4.6 of [2], we have $C(t) = TL\langle\sigma\rangle$ where $\sigma^2 = 1$ and TL is isomorphic to the centralizer of an involution in $GL(n + 2, 2)$. Suppose $C(t) \neq CT$. Then we can choose σ so that $C_L(\sigma) \cong \mathrm{Sp}(2n, 2)$. In particular, $O_2(C(\sigma) \cap C(t)) = \langle\sigma\rangle C_T(\sigma)$. Using $[T, b_1] \leqslant C_T(b_2)$, it is not difficult to show that $C_T(\sigma) \cap O_2(C(\sigma)) = 1$. The $P \times Q$ lemma applied to $\langle b_1\rangle \times \langle t\rangle$ acting on

$O_2(C(\sigma))$ then implies that b_1 centralizes $O_2(C(\sigma))$, contradicting the assumption that G has characteristic 2 type. Thus $C(t) = TL$ and $G \cong GL(n + 2, 2)$ by Suzuki [6].

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