THE BRUHAT ORDER OF THE SYMMETRIC GROUP
IS LEXICOGRAPHICALLY SHELLABLE

PAUL H. EDELMAN

Abstract. The title theorem is proven. It then follows from a theorem of Björner
that the simplicial complex of chains of this Bruhat order is shellable and thus
Cohen-Macaulay. It is further established that this complex is a double cone over a
triangulation of a sphere.

In this note we present an elementary proof that the Bruhat order of the
symmetric group $S_n$ is lexicographically shellable and hence Cohen-Macaulay.
Using a theorem of Verma we obtain as a corollary that $\Delta(S_n)$, the simplicial
complex of chains of $S_n$, is a double cone over a triangulation of a sphere of
dimension $(\frac{n}{2}) - 2$. We will employ the notation and terminology of Björner [2].

A finite poset $P$ is said to be bounded if it has a maximum and a minimum
element, denoted $\hat{1}$ and $\hat{0}$ respectively. It is called pure if all of its maximal chains
are the same length and it is graded if it is both bounded and pure. The rank of $P$ is
the length of a maximal chain. An element $x$ of a graded poset $P$ has a well-defined
rank $\rho(x)$ equal to the length of an unrefinable chain from $\hat{0}$ to $x$ in $P$. If $P$ is
bounded let $\tilde{P}$ be the poset $P - \{\hat{0}, \hat{1}\}$.

The order complex $\Delta(P)$ of a poset $P$ is the simplicial complex of all chains in $P$.
A poset is said to be shellable if $\Delta(P)$ is shellable. For the definition of a shellable
complex see [2] or [4]. Similarly $P$ is called Cohen-Macaulay if $\Delta(P)$ is. See [1], [2]

Let $C(P)$ be the set of covering relations

$$C(P) = \{ (x, y) \in P \times P \mid x \text{ is covered by } y \}.$$ 

An edge-labeling of $P$ is a map $\lambda: C(P) \rightarrow \Lambda$ where $\Lambda$ is some poset. An edge-labeling
is an assignment of elements of $\Lambda$ to the edges in the Hasse
diagram of $P$. An unrefinable chain $x_0 < x_1 < \cdots < x_n$ in a poset with an
edge-labeling $\lambda$ will be called increasing if $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < 
\lambda(x_{n-1}, x_n)$ in $\Lambda$.

With every saturated chain $c$, say with elements $x_0 < x_1 < \cdots < x_n$ of a poset
$P$ having an edge-labeling $\lambda$, we associate the $n$-tuple

$$\pi(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{n-1}, x_n)).$$

We call $\pi(c)$ the Jordan-Hölder (J-H) sequence of $c$. Totally order $\Lambda^n$ by the
lexicographic order: $(a_1, a_2, \ldots, a_n)$ precedes $(b_1, \ldots, b_n)$ if and only if $a_i < b_i$ in
the first coordinate where they differ.

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Let \( \lambda \) be the edge-labeling of a graded poset \( P \). \( \lambda \) is said to be an \textit{L-labeling} if it satisfies the following two conditions:

(i) In every interval \([x, y]\) of \( P \) there is a unique increasing unrefinable chain \( c \), \( x = x_0 < x_1 < \cdots < x_n = y \).

(ii) The J-H sequence of the unique chain from (i) is lexicographically first among the J-H sequences of all unrefinable chains \( x = z_0 < z_1 < \cdots < z_n = y \) in \([x, y]\).

A graded poset is called \textit{lexicographically shellable} if there exists an \( L \)-labeling of \( P \).

\textbf{Theorem (Björner [2]).} If \( P \) is lexicographically shellable then \( P \) is shellable and hence Cohen-Macaulay. \( \square \)

We now define the Bruhat order on the symmetric group \( S_n \). For our purposes \( S_n \) will be the set of all permutations of the set \([n] = \{1, 2, \ldots, n\}\). We will write \( \pi \in S_n \) as a word \( a_1a_2 \cdots a_n \) in the letters \( 1, 2, \ldots, n \). A \textit{reduction} of \( \pi \) is a permutation obtained from \( \pi \) by interchanging some \( a_i \) with some \( a_j \) provided \( i < j \) and \( a_i > a_j \). Define \( \sigma < \pi \) if \( \sigma \) can be obtained from \( \pi \) by a sequence of reductions. Figure 1 is a drawing of the poset \( S_3 \).

![Figure 1](image_url)

It is well known that the rank of a permutation \( \pi \) in \( S_n \) is the number of inversions in \( \pi \), i.e. the number of pairs \((i, j)\) where \( i < j \) and \( a_i > a_j \). Thus if \( \sigma \) is covered by \( \pi \) then \( \pi \) has one more inversion than \( \sigma \). The rank of \( S_n \) is \( \binom{n}{2} \).

\textbf{Theorem.} \( S_n \) is \textit{lexicographically shellable}.

\textbf{Proof.} Let \( Z \) be the set of ordered pairs \((i, j) \in [n] \times [n]\) such that \( i < j \). Totally order \( Z \) by \((i, j) < (r, s)\) if \( i < r \) or if \( i = r \) and \( j < s \). Let \( \lambda: C(S_n) \to Z \) be the labeling

\[ \lambda(\sigma_1, \sigma_2) = (i, j) \]

if \( i \) and \( j \) are interchanged in \( \sigma_1 \) to obtain \( \sigma_2 \) and \( i < j \). For example in \( S_3 \) we have \( \lambda(123, 213) = (1, 2) \) and \( \lambda(213, 312) = (2, 3) \).

We proceed to show that \( \lambda \) is an \( L \)-labeling by first showing that in any interval \([x, y]\) the lexicographically first chain increases and then showing that there is a unique increasing chain in \([x, y]\).
We will show that the lexicographically first chain increases by contradiction. There are a number of cases to consider. We will present one and leave the others to the reader. Let \( c \) be the lexicographically first chain in \([x, y]\), \( x = \pi_0 < \pi_1 < \cdots < \pi_n = y \) and suppose it has a decrease. Then there are three permutations \( \pi_{r-1} < \pi_r < \pi_{r+1} \) such that

\[
\lambda(\pi_{r-1}, \pi_r) > \lambda(\pi_r, \pi_{r+1}).
\]

Suppose \( \lambda(\pi_{r-1}, \pi_r) = (i, j) \) and \( \lambda(\pi_r, \pi_{r+1}) = (i, k) \) where \( j > k \). Then the permutation \( \pi_{r-1} \) looks like

\[
a_1 a_2 \ldots i \ldots j \ldots k \ldots a_n
\]

since the interchange \((i, j)\) must produce a cover. Define \( \pi'_r \) by interchanging \( i \) and \( k \) in \( \pi_{r-1} \). \( \pi'_r \) covers \( \pi_{r-1} \) and is covered by \( \pi_{r+1} \). Moreover \( \lambda(\pi_{r-1}, \pi'_r) = (i, k) \). Since \( (i, k) < (i, j) \) the chain \( c' \) with \( \pi'_r \) replacing \( \pi_r \) in \( c \) is lexicographically earlier than \( c \). This is a contradiction.

There are other cases to consider when the labels \( \lambda(\pi_{r-1}, \pi_r) \) and \( \lambda(\pi_r, \pi_{r+1}) \) are disjoint. They are similar to the above argument.

What is left to show is that the increasing chain in the interval \([x, y]\) is unique. This follows from a series of remarks. Let \( x = a_1 a_2 \ldots a_n \) and \( y = b_1 b_2 \ldots b_n \). Define \( p(r) = j \) if and only if \( r \sim c \) and define \( s(r) = j \) if and only if \( r = b \). Let \( i \) be the smallest number such that \( p(i) \neq s(i) \).

**Remark 1.** No number less than \( i \) will appear in any label in \([x, y]\). Suppose this were not true. Then there is some label \((j, k)\) in \([x, y]\) where \( j < i \) and \( j \) is the smallest number appearing in such a label. Since \( p(j) = s(j) \) when \( j \) is interchanged with \( k \), \( j \) is moved to the right of the correct position for it in \( y \). Since no smaller number appears as a label \( j \) cannot be moved back to the left. Hence \( j \) cannot appear in a label in \([x, y]\).

**Remark 2.** \( p(i) < s(i) \). This follows from an argument similar to that used in Remark 1.

**Remark 3.** In an increasing chain, the first label contains \( i \). The element \( i \) must be switched sometime to get from \( x \) to \( y \) and since it is the smallest number it must occur first.

**Remark 4.** If \( \lambda(x, \pi_1) = (i, k) \) where \( \pi_1 \) is the second permutation in an increasing chain, then \( p(k) < s(i) \). This follows from the same arguments as those used in Remark 1.

Let \( j \) be the smallest number such that \( i < j \) and \( p(i) < p(j) < s(i) \).

**Remark 5.** The first label on an increasing chain is \((i, j)\). Suppose this were not the case. By Remark 3 the first label involves \( i \). Suppose it is \((i, k) \), \( k \neq j \). By Remark 4, \( p(k) < s(i) \). If \( p(j) < p(k) \) then the switch \((i, k)\) increases the number of inversions in the permutation by at least two, since \( k > j > i \). So \((i, k)\) does not produce a cover. Hence \( p(k) < p(j) \). But sometime \( i \) and \( j \) must switch, since \( p(i) < p(k) < p(j) \). Then the label \((i, j)\) appears which forces a decrease. So the first label must be \((i, j)\).

Hence an increasing chain is uniquely determined. Since the lexicographically first chain increases, \( \lambda \) is an \( L \)-labeling and the proof is complete. \( \square \)
A graded poset is called Eulerian if in every interval \([x, y]\) the identity
\[
\sum_{x < z < y} (-1)^{p(z) - p(x)} = 0
\]
holds. If \(P\) is Eulerian then \(\bar{P}\) also satisfies \((*)\) for all intervals \([x, y]\). It is easily seen that every interval of rank 2 in an Eulerian poset is isomorphic to the poset in Figure 2. R. Stanley observed that if \(P\) satisfies \((*)\) and is of rank \(k\) then \(\Delta(P)\) is a pseudomanifold of dimension \(k\), i.e. every \(k - 1\) face is contained in exactly two \(k\) faces. From this observation we deduce

**Corollary.** \(\Delta(\bar{S}_n)\) is a triangulation of a sphere of dimension \(\left(\begin{smallmatrix} n-1 \end{smallmatrix}\right) - 2\).

**Proof.** Verma [7] has shown that \(S_n\) is Eulerian. Hence \(\Delta(\bar{S}_n)\) is a pseudomanifold. Since \(\Delta(\bar{S}_n)\) is shellable by the previous theorem so is \(\Delta(\bar{S}_n)\). Since it is known that a shellable pseudomanifold is a sphere (see for example [3, p. 444]) the corollary is proven.

R. Proctor has extended the Theorem to the Bruhat orders of the other classical Weyl groups as well as their quotients by parabolic subgroups [5].

**Figure 2**

Note added in proof. Björner and Wachs have recently generalized the Theorem for all Coxeter groups modulo a parabolic subgroup ([Bruhat orders of Coxeter groups and shellability, Report 1980-No. 20, Department of Mathematics, University of Stockholm]).

**References**

5. R. Proctor, Classical Bruhat orders are lexicographically shellable (in preparation).