PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 3, July 1981

THE BRUHAT ORDER OF THE SYMMETRIC GROUP IS LEXICOGRAPHICALLY SHELLABLE

PAUL H. EDELMAN

ABSTRACT. The title theorem is proven. It then follows from a theorem of Björner that the simplicial complex of chains of this Bruhat order is shellable and thus Cohen-Macaulay. It is further established that this complex is a double cone over a triangulation of a sphere.

In this note we present an elementary proof that the Bruhat order of the symmetric group S_n is lexicographically shellable and hence Cohen-Macaulay. Using a theorem of Verma we obtain as a corollary that $\Delta(S_n)$, the simplicial complex of chains of S_n, is a double cone over a triangulation of a sphere of dimension $\binom{n}{2} - 2$. We will employ the notation and terminology of Björner [2].

A finite poset P is said to be *bounded* if it has a maximum and a minimum element, denoted $\hat{1}$ and $\hat{0}$ respectively. It is called *pure* if all of its maximal chains are the same length and it is *graded* if it is both bounded and pure. The *rank* of P is the length of a maximal chain. An element x of a graded poset P has a well-defined rank $\rho(x)$ equal to the length of an unrefinable chain from $\hat{0}$ to x in P. If P is bounded let $\bar{P}$ be the poset $P - \{\hat{0}, \hat{1}\}$.

The order complex $\Delta(P)$ of a poset P is the simplicial complex of all chains in P. A poset is said to be *shellable* if $\Delta(P)$ is shellable. For the definition of a shellable complex see [2] or [4]. Similarly P is called *Cohen-Macaulay* if $\Delta(P)$ is. See [1], [2] or [6] for the definition and significance of a Cohen-Macaulay complex.

Let $C(P)$ be the set of covering relations

$$C(P) = \{(x, y) \in P \times P \mid x \text{ is covered by } y\}.$$

An *edge-labeling* of P is a map $\lambda: C(P) \to \Lambda$ where Λ is some poset. An edge-labeling corresponds to an assignment of elements of Λ to the edges in the Hasse diagram of P. An unrefinable chain $x_0 < x_1 < \cdots < x_n$ in a poset with an edge-labeling λ will be called *increasing* if $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{n-1}, x_n)$ in Λ.

With every saturated chain c, say with elements $x_0 < x_1 < \cdots < x_n$ of a poset P having an edge-labeling λ, we associate the n-tuple

$$\pi(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{n-1}, x_n)).$$

We call $\pi(c)$ the *Jordan-Hölder* (J-H) *sequence* of c. Totally order Λ^n by the lexicographic order: $(a_1, a_2, \ldots, a_n)$ precedes $(b_1, \ldots, b_n)$ if and only if $a_i < b_i$ in the first coordinate where they differ.

Received by the editors May 30, 1980.
1980 *Mathematics Subject Classification*. Primary 06A10.

Let λ be the edge-labeling of a graded poset P. λ is said to be an *L-labeling* if it satisfies the following two conditions:

(i) In every interval $[x, y]$ of P there is a unique increasing unrefinable chain c, $x = x_0 < x_1 < \cdots < x_n = y$.

(ii) The J-H sequence of the unique chain from (i) is lexicographically first among the J-H sequences of all unrefinable chains $x = z_0 < z_1 < \cdots < z_n = y$ in $[x, y]$.

A graded poset is called *lexicographically shellable* if there exists an L-labeling of P.

THEOREM (BJÖRNER [2]). *If P is lexicographically shellable then P is shellable and hence Cohen-Macaulay.* □

We now define the Bruhat order on the symmetric group S_n. For our purposes S_n will be the set of all permutations of the set $[n] = \{1, 2, \ldots, n\}$. We will write $\pi \in S_n$ as a word $a_1 a_2 \ldots a_n$ in the letters $1, 2, \ldots, n$. A *reduction* of π is a permutation obtained from π by interchanging some a_i with some a_j provided $i < j$ and $a_i > a_j$. Define $\sigma \leq \pi$ if σ can be obtained from π by a sequence of reductions. Figure 1 is a drawing of the poset S_3.

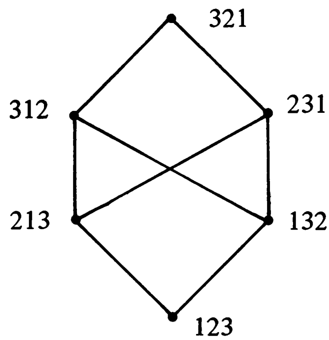

FIGURE 1

It is well known that the rank of a permutation π in S_n is the number of inversions in π, i.e. the number of pairs (i, j) where $i < j$ and $a_i > a_j$. Thus if σ is covered by π then π has one more inversion than σ. The rank of S_n is $\binom{n}{2}$.

THEOREM. *S_n is lexicographically shellable.*

PROOF. Let Z be the set of ordered pairs $(i, j) \in [n] \times [n]$ such that $i < j$. Totally order Z by $(i, j) < (r, s)$ if $i < r$ or if $i = r$ and $j < s$. Let λ: $C(S_n) \to Z$ be the labeling

$$\lambda(\sigma_1, \sigma_2) = (i, j)$$

if i and j are interchanged in σ_1 to obtain σ_2 and $i < j$. For example in S_3 we have $\lambda(123, 213) = (1, 2)$ and $\lambda(213, 312) = (2, 3)$.

We proceed to show that λ is an L-labeling by first showing that in any interval $[x, y]$ the lexicographically first chain increases and then showing that there is a unique increasing chain in $[x, y]$.

We will show that the lexicographically first chain increases by contradiction. There are a number of cases to consider. We will present one and leave the others to the reader. Let c be the lexicographically first chain in $[x, y]$, $x = \pi_0 < \pi_1 < \cdots < \pi_n = y$ and suppose it has a decrease. Then there are three permutations $\pi_{r-1} < \pi_r < \pi_{r+1}$ such that

$$\lambda(\pi_{r-1}, \pi_r) > \lambda(\pi_r, \pi_{r+1}).$$

Suppose $\lambda(\pi_{r-1}, \pi_r) = (i, j)$ and $\lambda(\pi_r, \pi_{r+1}) = (i, k)$ where $j > k$. Then the permutation π_{r-1} looks like

$$a_1 a_2 \ldots i \ldots j \ldots k \ldots a_n$$

since the interchange (i, j) must produce a cover. Define π_r' by interchanging i and k in π_{r-1}. π_r' covers π_{r-1} and is covered by π_{r+1}. Moreover $\lambda(\pi_{r-1}, \pi_r') = (i, k)$. Since $(i, k) < (i, j)$ the chain c' with π_r' replacing π_r in c is lexicographically earlier than c. This is a contradiction.

There are other cases to consider when the labels $\lambda(\pi_{r-1}, \pi_r)$ and $\lambda(\pi_r, \pi_{r+1})$ are disjoint. They are similar to the above argument.

What is left to show is that the increasing chain in the interval $[x, y]$ is unique. This follows from a series of remarks. Let $x = a_1 a_2 \ldots a_n$ and $y = b_1 b_2 \ldots b_n$. Define $p(r) = j$ if and only if $r = a_j$ and define $s(r) = j$ if and only if $r = b_j$. Let i be the smallest number such that $p(i) \neq s(i)$.

REMARK 1. No number less than i will appear in any label in $[x, y]$. Suppose this were not true. Then there is some label (j, k) in $[x, y]$ where $j < i$ and j is the smallest number appearing in such a label. Since $p(j) = s(j)$ when j is interchanged with k, j is moved to the right of the correct position for it in y. Since no smaller number appears as a label j cannot be moved back to the left. Hence j cannot appear in a label in $[x, y]$.

REMARK 2. $p(i) < s(i)$. This follows from an argument similar to that used in Remark 1.

REMARK 3. In an increasing chain, the first label contains i. The element i must be switched sometime to get from x to y and since it is the smallest number it must occur first.

REMARK 4. If $\lambda(x, \pi_1) = (i, k)$ where π_1 is the second permutation in an increasing chain, then $p(k) \leqslant s(i)$. This follows from the same arguments as those used in Remark 1.

Let j be the smallest number such that $i < j$ and $p(i) < p(j) \leqslant s(i)$.

REMARK 5. The first label on an increasing chain is (i, j). Suppose this were not the case. By Remark 3 the first label involves i. Suppose it is (i, k), $k \neq j$. By Remark 4, $p(k) \leqslant s(i)$. If $p(j) < p(k)$ then the switch (i, k) increases the number of inversions in the permutation by at least two, since $k > j > i$. So (i, k) does not produce a cover. Hence $p(k) < p(j)$. But sometime i and j must switch, since $p(i) < p(k) < p(j)$. Then the label (i, j) appears which forces a decrease. So the first label must be (i, j).

Hence an increasing chain is uniquely determined. Since the lexicographically first chain increases, λ is an L-labeling and the proof is complete. $\square$

A graded poset is called *Eulerian* if in every interval $[x, y]$ the identity

$$\sum_{x \leq z \leq y} (-1)^{\rho(z)-\rho(x)} = 0 \qquad (*)$$

holds. If P is Eulerian then $\bar{P}$ also satisfies $(*)$ for all intervals $[x, y]$. It is easily seen that every interval of rank 2 in an Eulerian poset is isomorphic to the poset in Figure 2. R. Stanley observed that if P satisfies $(*)$ and is of rank k then $\Delta(P)$ is a pseudomanifold of dimension k, i.e. every $k - 1$ face is contained in exactly two k faces. From this observation we deduce

COROLLARY. *$\Delta(\bar{S}_n)$ is a triangulation of a sphere of dimension* $\binom{n}{2} - 2$.

PROOF. Verma [7] has shown that S_n is Eulerian. Hence $\Delta(\bar{S}_n)$ is a pseudomanifold. Since $\Delta(S_n)$ is shellable by the previous theorem so is $\Delta(\bar{S}_n)$. Since it is known that a shellable pseudomanifold is a sphere (see for example [3, p. 444]) the corollary is proven.

R. Proctor has extended the Theorem to the Bruhat orders of the other classical Weyl groups as well as their quotients by parabolic subgroups [5].

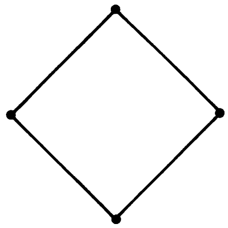

FIGURE 2

NOTE ADDED IN PROOF. Björner and Wachs have recently generalized the Theorem for all Coxeter groups modulo a parabolic subgroup (*Bruhat orders of Coxeter groups and shellability*, Report 1980-No. 20, Department of Mathematics, University of Stockholm).

REFERENCES

1. K. Baclawski, *Cohen-Macaulay ordered sets*, J. Algebra **63** (1980), 226–258.

2. A. Björner, *Shellable and Cohen-Macaulay partially ordered sets*, Trans. Amer. Math. Soc. **260** (1980), 159–183.

3. G. Danaraj and V. Klee, *Shellings of spheres and polytopes*, Duke Math. J. **41** (1974), 443–451.

4. ______, *Which spheres are shellable?*, Algorithmic Aspects of Combinatorics, Ann. Discrete Math. **2** (1978), 33–52.

5. R. Proctor, *Classical Bruhat orders are lexicographically shellable* (in preparation).

6. R. Stanley, *Cohen-Macaulay complexes*, Higher Combinatorics, edited by M. Aigner, Reidel, Dordrecht and Boston, Mass., 1977, pp. 51–62.

7. D.-N. Verma, *Möbius inversion for the Bruhat ordering on a Weyl group*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 393–399.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Current address: Department of Mathematics E1, University of Pennsylvania, Philadelphia, Pennsylvania 19104