EQUIVALENCE OF CERTAIN REPRESENTING MEASURES

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Abstract. If an interior component \( \Omega \) of a compact \( K \subset \mathbb{C} \) is a part for \( R(K) \), then given \( z_1, z_2 \) in \( \Omega \) and a representing measure \( \lambda_1 \) for \( z_1 \) there is a representing measure for \( z_2 \) equivalent to \( \lambda_1 \).

Given two points \( \varphi \) and \( \psi \) in the same Gleason part of a uniform algebra, and \( \lambda \) in \( M_\varphi \) (the set of representing measures for \( \varphi \)), a well-known result of Bishop [2, p. 143] insures there is a \( \mu \) in \( M_\psi \) which bounds \( \lambda \), and indeed there is a pair \( \lambda, \mu \in M_\varphi \) which mutually bound one another. However it is not in general the case that each \( \lambda \) in \( M_\varphi \) is equivalent to some element of \( M_\psi \), and our purpose is to point out one instance where that occurs.

Recall that for \( K \subset \mathbb{C} \) compact, \( R(K) \) is the uniform closure in \( C(K) \) of the rational functions.

Theorem. Suppose \( K \subset \mathbb{C} \) is compact and \( \Omega \) is a component of the interior \( K^\circ \) which is also a Gleason part for \( R(K) \). Then for \( z_1, z_2 \in \Omega \) and \( \lambda_1 \in M_{z_1} \) there is a \( \lambda_2 \in M_{z_2} \) equivalent to \( \lambda_1 \); moreover for any nonzero measure \( \mu \) on \( \partial K \) orthogonal to \( R(K) \) with \( \mu \ll \lambda_1 \) there is a \( \lambda \in M_{z_2} \) equivalent to \( \mu \).

Here \( M_z \) is the set of representing measures on \( \partial K \), as usual. Of course the first conclusion fails if the part containing the component \( \Omega \) contains a boundary point since no measure representing \( z \in \Omega \) is equivalent to a point mass; and it could only extend to a part consisting of several interior components if these all shared the same boundary (since \( \partial \Omega \) is always the topological support of harmonic measure for \( z \in \Omega \)). Consequently it seems unlikely that it holds for any noncomponent parts.

It is worth noting that the result is not due to all elements of \( M_{z_1} \) being mutually equivalent; an example is provided by the champagne bubble set \( K \) (given in [5, 27.6]) built from Beurling’s function by McKissick in constructing his regular uniform algebra: there \( K \) consists of the unit disc \( D \) less disjoint subdiscs of finite total circumference converging to \( \partial D \), \( K^\circ \) is dense and connected, and there is an \( f \in R(K) \) with \( f^{-1}(0) = \partial D \). Consequently for \( z \in K^\circ \), \( \partial D \) is necessarily a Jensen null set (so of harmonic measure \( \lambda_2 \) zero); but via Cauchy \( dz \) provides an orthogonal measure \( \mu \) on \( \partial K \), which is absolutely continuous with respect to some \( \lambda \in M_{z_1} \) by the known decomposition of orthogonal measures [5, 23.6] and Wilken’s theorem [2, p. 47], and \( \lambda \) and \( \lambda_2 \) are inequivalent.

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Our result is a simple consequence of an abstract F. and M. Riesz theorem for bands due to Brian J. Cole (unpublished) and König and Seever [4] which we state in a form convenient for our use (cf. [1, 3.1, p. 31]) for a uniform algebra $A$ on $X$ ($= \partial K$ in our application).

**Theorem (Cole-König-Seever).** Let $m$ be a probability measure on $X$ and suppose $M_\varphi(m) = M_\varphi \cap L^1(m) \neq \emptyset$. If $\mu \ll m$ is a measure orthogonal to $A$ and we choose $\lambda$ in $M_\varphi(m)$ so that $\|\mu_\lambda\|$ is a maximum, where $\mu_\lambda$ is the component of $\mu$ absolutely continuous with respect to $\lambda$, then $\mu_\lambda \perp A$.

(Here $L^1(m)$ is our band [1], and $\mu = \mu_\lambda + \mu_\lambda'$, where $\mu_\lambda'$ is the component of $\mu$ singular with respect to $\lambda$, is our band decomposition of $\mu$.)

We first apply this F. and M. Riesz theorem to the uniform algebra setting. (I am indebted to the referee for considerable simplification in what follows.)

**Proposition.** Let $B$ be a uniform algebra on $X$, and let $g \in B$ be nonvanishing on $X$, with $1/g \in B$ while $C(X) = \{B, 1/g\}$, the closed algebra generated by $B$ and $1/g$. Trivially then $A = C + gB$ is a closed subalgebra of $C(X)$ with $gB$ a maximal ideal, the kernel of a multiplicative linear functional $\varphi$ on $A$.

Suppose $\mu \perp B$, $\mu \neq 0$. Then there is a representing measure $\lambda$ for $\varphi$ which is equivalent to $|\mu|$.

To begin, since $\mu \neq 0$ and $C(X) = \{B, 1/g\}$, there is a least integer $n > 1$ with $\mu \perp g^{-n}B$, so $g^{-n}\mu \perp gB$ while $g^{-n}\mu(f) = 1$ for some $f \in B$. Thus $fg^{-n}\mu$ is a complex measure representing $\varphi$ on $A = C + gB$, and so dominates a true (i.e., $> 0$) representing measure $\lambda$ for $\varphi$ [2, p. 33], and $\lambda \in M_\varphi(|\mu|) = M_\varphi \cap L^1(|\mu|)$.

Now in $M_\varphi(|\mu|)$ we can find an element $\lambda$ which is maximal in the sense that for any other element $\lambda'$ we have $\lambda' \ll \lambda$, and this dominating $\lambda$ clearly has the property that it maximizes $\|(bu)_\lambda\|$ for any fixed bounded function $b$, where $(bu)_\lambda$ is the component of $bu \ll \lambda$. So for $b \in B$ if we apply the Cole-König-Seever theorem to the measure $bu \perp A$ we have $(bu)_\lambda' = b\mu_\lambda' \perp A$, where the prime indicates the $\lambda$-singular component. In particular $\mu_\lambda'(b) = 0$ for all $b \in B$, so $\mu_\lambda' \perp B$.

But if $\mu_\lambda' \neq 0$, as in the first paragraph we obtain a representing measure $\lambda'$ for $\varphi$ on $A$ with $\lambda' \ll |\mu_\lambda'|$, hence singular with respect to $\lambda$, and clearly $(\mu_\lambda')_\lambda'$, the component of $\mu_\lambda'$, is nonzero; since

$$\|\mu(\lambda + \lambda')/2\| > \|\mu_\lambda\| + \|(\mu_\lambda')_\lambda'\| > \|\mu_\lambda\|$$

contradicts the maximality of $\|\mu_\lambda\|$ we conclude $\mu_\lambda' = 0$. Thus $\mu = \mu_\lambda \ll \lambda$, and $\lambda$ and $|\mu|$ are equivalent as asserted.

**Corollary.** Let $K \subset C$ be compact and suppose the interior $K^\circ$ is connected and dense in $K$. Let $\mu$ be a measure on $\partial K$ with $\mu \perp R(K)$, and let $z \in K^\circ$. Then there is a representing measure $\lambda$ for $z$ on $R(K)$ which is equivalent to $|\mu|$.

Here we apply the proposition to $X = \partial K$ and the uniform algebra $B = R(K)$ on $X$, with $g(\xi) = \xi - z$. We have $A = R(K) = B$ and $\varphi$ evaluation at $z$, and by

\[\text{If } \|\mu_\lambda\| \to \text{maximum then } \lambda = \sum_1^\infty 2^{-n}\mu_\lambda \text{ provides such a maximizing } \lambda.\]
Runge’s theorem $[R(K), 1/g] = R(\partial K)$. Since every point of $\partial K$ lies in the closure of the connected interior $K^o$ we know it is a peak point for $R(\partial K)$, and so $R(\partial K) = C(\partial K)$ by Bishop’s theorem [2, p. 54]; thus the proposition applies to yield its corollary.

We can now obtain our theorem from the corollary. Note that its first assertion follows from the second by setting $\mu = (g - z_0)\lambda_z$; so we only need to see any $\mu$ on $\partial K$ orthogonal to $R(K)$ with $\mu$ absolutely continuous with respect to a representing measure $\lambda_0$ for some point in $\Omega$ is equivalent, for any $z$ in $\Omega$, to some $\lambda \in M_A$. But our $\mu$ necessarily has, for $z \notin \Omega^{-}$, a Cauchy transform [2, p. 46] $\hat{\mu}(z)$ which must vanish: if $\hat{\mu}(z) \neq 0$ for $z \notin \Omega^{-}$ then $|\mu|$ dominates some representing measure $\lambda$ for $z$, by an old result of Bishop, so since $\lambda_0$ and $\lambda$ are not mutually singular, $z$ would lie in the part $\Omega$. Since $\hat{\mu}(z) = 0$ for all $z \notin \Omega^{-}$ implies $\mu \perp R(\Omega^{-})$, while $\Omega^{-}$ necessarily has its interior a union of components of $K^o$, hence just $\Omega$, for $z \in \Omega$ we can apply the corollary to $\Omega^{-}$ to obtain a representing measure $\lambda$ for $z$ on $R(\Omega^{-})$ equivalent to $|\mu|$. Trivially $\lambda$ represents $z$ on $R(K)$, so we are done.

In fact the theorem holds with $R(K)$ replaced by $A(K)$, or any $T$-invariant closed subalgebra [3] $A_0$ of $C(K)$ lying between $R(K)$ and $A(K)$; the proof is essentially the same since, for such an algebra, $A_0 = C + (z - z_0)A_0$ (for $z_0 \in \Omega$, while $K$ is the spectrum and $\partial K$ the Silov boundary. (In place of applying the corollary one can appeal to the proposition applied to $B = (A_1\Omega^{-})^{-}= C + gB$, $g(\xi) = \xi - z$, noting that $[B, 1/g] \supset R(\partial \Omega) = C(\partial \Omega).$)

REFERENCES


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