ON THE DINI TEST AND DIVERGENCE OF FOURIER SERIES

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Abstract. In this note we prove that no condition weaker than the Dini assures the pointwise convergence of a Fourier series in a set of positive measure.

1. Statement of results. It is well known that if a function \( f \) belonging to \( L^1[0, 2\pi] \) satisfies at a point \( x \) the condition

\[
\int_{-\delta}^{\delta} \frac{|f(x) - f(x - t)|}{|t|} \, dt < \infty
\]

(1.1)

then the partial sums \( S_m \) of the Fourier series of \( f \) converge to \( f(x) \).

The above condition is optimal for individual points. In fact, “Given any continuous \( \mu(t) > 0 \) such that \( \mu(t)/t \) is not integrable in a neighborhood of \( t = 0 \), we can find a continuous function \( f \), such that \( |f(t) - f(0)| \leq \mu(t) \) for small \( t \) and \( S_m(f) \) diverges at \( t = 0 \).” (See [5, Vol. I, Theorem 2.4, p. 303].)

On the other hand, if a periodic function \( f \) satisfies the conditions \( |f(x + t) - f(x)| < \mu(t) \), \( 0 < t < \delta(x) \), in a set \( E \) of positive measure, then \( S_m(f) \) converges at almost every \( x \) in \( E \). This fact is an easy consequence of Carleson’s theorem (see [1]).

The purpose of this note is to show that the condition (1.1) is the optimal one for \( x \) belonging to a set \( E \) of positive measure, if we give a suitable definition of what we understand by a weaker condition. More precisely, for each \( w(t) \) that satisfies

\[
\begin{align*}
(i) & \quad w(t) \text{ increasing and continuous}, \quad w(0) = 0, \\
(ii) & \quad \int_{0}^{\delta} w(t) \, dt/t = \infty,
\end{align*}
\]

(1.2)

we have

2. Theorem. There is a function \( g \) belonging to \( L^1[0, 2\pi] \) and a set \( F \) of positive measure such that

\( (a) \) \( \int_{-\delta}^{\delta} \frac{|g(x + t) - g(x)|}{|t|} \, w(|t|)/|t| \, dt < \infty \), \( x \in F \).

\( (a\alpha) \) The partial sums \( S_n(g) \) of the Fourier series of \( g \) diverge a.e. in \( F \).

Proof. First, we invoke the following theorem due to Marcinkiewicz (see [3]):

Theorem 2.1. Let \( \varphi(t) \) be a continuous function, increasing and such that \( \varphi(0) = 0 \). Assume in addition that

\( (i) \) \( [\varphi(t)]^{-1} = o(\log 1/t) \), \( t \to 0 \).

Then, there is a function \( f \in L^1[0, 2\pi] \) such that

\( (\beta) \) \( (1/|h|) \int_{-\delta}^{\delta} |f(x + t) - f(x)| \, dt = O[\varphi(|h|)] \) a.e.;

\( (\beta\beta) \) The partial sums \( S_n(f) \) of the Fourier series of \( f \) diverge a.e.
Secondly, we need a lemma of very well-known type.

**Lemma 2.2.** Let \( \varphi(t) \) be defined by

\[
\varphi(t) = \left[ \int_{t}^{1} w(s) \frac{ds}{s} \right]^{-1}, \quad 0 < t < 1,
\]

and \( w(s) \) as in (1.2). Suppose that a function \( f \in L^1[0, 2\pi] \) satisfies

(i) \( (1/|h|) \int_{h}^{0} |f(x + t) - f(x)| \, dt = O(\varphi(|h|)) \) a.e. in \( [0, 2\pi] \).

Then, for each \( \epsilon > 0 \) there are a perfect subset \( F \) of \( [0, 2\pi] \) and a constant \( C \) such that

(\( \alpha \)) \( |F| > 2^n - \epsilon \),

(\( \alphaa \)) \( |f(x_1) - f(x_2)| < C\varphi(|x_1 - x_2|) \) whenever \( x_1, x_2 \in F \) and \( 0 < |x_1 - x_2| < 1/2 \),

(\( \alphaaa \)) \( (1/|h|) \int_{h}^{0} |f(x) - f(x + t)| \, dt < C\varphi(|h|) \) whenever \( x \in F \).

The proof follows verbatim the lines of the corresponding result in [5, Vol. II, p. 171]. (There, \( \varphi(t) = 1/|\log t| \).) One also has to use the fact

\[
\varphi(ct) < k\varphi(t), \quad c > 1, \quad 0 < t < 1/4C, \quad (2.2.1)
\]

which is an easy consequence of the particular form that \( \varphi(t) \) has.

3. Final steps. Consider

\[
\varphi(t) = \left[ \int_{t}^{1} \tilde{w}(s) \frac{ds}{s} \right]^{-1}, \quad 0 < t < 1/2,
\]

where \( \tilde{w}(s) = \max(|\log s|^{-\delta}, w(s)), 0 < \delta < \frac{1}{2}, 0 < s < \frac{1}{2} \) and \( w(s) \) as in (1.2). On account of the particular form of \( \varphi(t) \), we get

\[
[\varphi(t)]^{-1} = o(\log(1/t)), \quad t \to 0. \quad (3.1)
\]

Let \( \tilde{f}(x) \) be the function whose existence is assured by Theorem 2.1 for the particular choice we made of \( \varphi(t) \). Let \( \epsilon > 0 \) be a positive constant, \( F \) and \( C \) be the perfect subset and the constant of Lemma 2.2. Call \( \tilde{f} \) any continuous extension of \( f \) from \( F \) to \( [0, 2\pi] \) such that

\[
|\tilde{f}(x_1) - \tilde{f}(x_2)| < C\varphi(|x_1 - x_2|),
\]

\[ x_1, x_2 \in [0, 2\pi] \text{ and } 0 < |x_1 - x_2| < 1/2. \quad (3.2)\]

Write now \( f = \tilde{f} + g \) and consider

\[
\int_{F} \left( \int_{0}^{2\pi} |g(x) - g(y)| \frac{\tilde{w}(|x - y|)}{|x - y|} \, dy \right) \, dx. \quad (3.3)
\]

Since \( g(x) = 0, x \in F \), it is enough to estimate

\[
\int_{F} \left( \int_{G} |g(y)| \frac{\tilde{w}(|x - y|)}{|x - y|} \, dy \right) \, dx. \quad (3.4)
\]

Here, \( G \) denotes the complement of \( F \) with respect to \( [0, 2\pi] \). Write \( G = \bigcup_{i=1}^{\infty} I_k \), \( I_i \cap I_j = \emptyset, i \neq j \), where \( \text{dist}(I_k, F) \) satisfies

\[
|I_k| < \text{dist}(I_k, F) < 2|I_k|, \quad k = 1, 2, \ldots. \quad (3.5)
\]
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On account of the definition of \( \tilde{f} \) and the above inequality we have

\[
\int_{I_k} |g(y)| \, dy < K_0 \varphi(|I_k|)|I_k|, \tag{3.6}
\]

which is valid for all \( k \) and a suitable choice of \( K_0 \).

Interchanging the order of integration in (3.4) and using (3.6) we get

\[
\sum_{1}^{\infty} \int_{I_k} |g(y)| \left( \int_{F} \frac{\overline{w}(x - y)}{|x - y|} \, dx \right) \, dy < \text{(constant)} \sum_{1}^{\infty} \left( \int_{I_k} |g(y)| \, dy \right) \varphi(|I_k|)^{-1}
\]

\[
< \text{constant} \sum_{1}^{\infty} |I_k|. \tag{3.7}
\]

The finiteness of (3.3) shows that

\[
\int_{-\delta}^{\delta} |g(x)| - g(x + t)| \overline{w}(|t|) \, dt < \infty \tag{3.8}
\]

for almost every \( x \) in \( F \). By construction the Fourier series of \( \tilde{f} \) diverges a.e. The Fourier series of \( f \) converges a.e. because

\[
\int_{0}^{2\pi} \int_{0}^{2\pi} |\tilde{f}(x) - \tilde{f}(y)|^2 \frac{1}{|x - y|} \, dx \, dy < \infty. \tag{3.9}
\]

(By construction we have that \( \varphi(t) < C(1/|\log t|^{1-\delta}) \) with \( 0 < \delta < 1/4 \).) (See Theorem 1.14 in [5, p. 164].)

Finally, (3.8) above also holds for \( w(t) \) on account of the fact that \( w(t) < \overline{w}(t) \). This concludes the proof.

4. A few additional remarks. The Dini condition (1.1) for \( x \in E, |E| > 0 \), is not comparable to the condition

\[
\frac{1}{h} \int_{0}^{h} |f(x + t) - f(x)| \, dt = O\left(\frac{1}{|\log h|}\right), \quad x \in E. \tag{4.1}
\]

The reason for that is the fact that (1.1) assures convergence of \( S_m(f) \) everywhere in \( E \), while (4.1) assures only convergence a.e. in \( E \) (see [4] and [5, Vol. I, p. 302 (2.1)]).

Theorem 2.1 above was proved by Marcinkiewicz to show that condition (4.1) can not be replaced by any similar weaker one.

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REFERENCES


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