PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 3, July 1981

ON THE DINI TEST AND DIVERGENCE OF FOURIER SERIES

CALIXTO P. CALDERON[1]

ABSTRACT. In this note we prove that no condition weaker than the Dini assures the pointwise convergence of a Fourier series in a set of positive measure.

1. Statement of results. It is well known that if a function f belonging to $L^1[0, 2\pi]$ satisfies at a point x the condition

$$\int_{-\delta}^{\delta} \frac{|f(x) - f(x-t)|}{|t|}\, dt < \infty \tag{1.1}$$

then the partial sums S_m of the Fourier series of f converge to $f(x)$.

The above condition is optimal for individual points. In fact, "Given any continuous $\mu(t) \geqslant 0$ such that $\mu(t)/t$ is not integrable in a neighborhood of $t = 0$, we can find a continuous function f, such that $|f(t) - f(0)| \leqslant \mu(t)$ for small t and $S_m(f)$ diverges at $t = 0$." (See [5, Vol. I, Theorem 2.4, p. 303].) On the other hand, if a periodic function f satisfies the conditions $|f(x+t) - f(x)| < \mu(t)$, $0 < t < \delta(x)$, x in a set E of positive measure, then $S_m(f)$ converges at almost every x in E. This fact is an easy consequence of Carleson's theorem (see [1]).

The purpose of this note is to show that the condition (1.1) is the optimal one for x belonging to a set E of positive measure, if we give a suitable definition of what we understand by a weaker condition. More precisely, for each $w(t)$ that satisfies

$$\text{(i)}\quad w(t) \text{ increasing and continuous, } w(0) = 0,$$

$$\text{(ii)}\quad \int_0^{\delta} w(t)\, dt/t = \infty, \tag{1.2}$$

we have

2. THEOREM. *There is a function g belonging to $L^1[0, 2\pi]$ and a set F of positive measure such that*

(α) $\int_{-\delta}^{\delta} |g(x+t) - g(x)|\, w(|t|)/|t|\, dt < \infty$, $x \in F$.

($\alpha\alpha$) *The partial sums $S_n(g)$ of the Fourier series of g diverge a.e. in F.*

PROOF. First, we invoke the following theorem due to Marcinkiewicz (see [3]):

THEOREM 2.1. *Let $\varphi(t)$ be a continuous function, increasing and such that $\varphi(0) = 0$. Assume in addition that*

(i) $[\varphi(t)]^{-1} = o(\log 1/t)$, $t \to 0$.

Then, there is a function $f \in L^1[0, 2\pi]$ such that

(β) $(1/|h|) \int_0^h |f(x+t) - f(x)|\, dt = O[\varphi(|h|)]$ *a.e.*;

($\beta\beta$) *The partial sums $S_n(f)$ of the Fourier series of f diverge a.e.*

Received by the editors July 23, 1980.
1980 *Mathematics Subject Classification*. Primary 42A20.
[1]Research supported by an N.S.F. grant.

Secondly, we need a lemma of very well-known type.

LEMMA 2.2. *Let* $\varphi(t)$ *be defined by*

$$\varphi(t) = \left[\int_t^1 w(s)\frac{ds}{s}\right]^{-1}, \qquad 0 < t < 1,$$

and $w(s)$ *as in* (1.2). *Suppose that a function* $f \in L^1[0, 2\pi]$ *satisfies*

(i) $(1/|h|)\int_0^h |f(x+t) - f(x)|\,dt = O[\varphi(|h|)]$ *a.e. in* $[0, 2\pi]$.

Then, for each $\varepsilon > 0$ *there are a perfect subset* F *of* $[0, 2\pi]$ *and a constant* C *such that*

(α) $|F| > 2\pi - \varepsilon$,

($\alpha\alpha$) $|f(x_1) - f(x_2)| < C\varphi(|x_1 - x_2|)$ *whenever* $x_1, x_2 \in F$ *and* $0 < |x_1 - x_2| < 1/2$,

($\alpha\alpha\alpha$) $(1/|h|)\int_0^h |f(x) - f(x+t)|\,dt < C\varphi(|h|)$ *whenever* $x \in F$.

The proof follows verbatim the lines of the corresponding result in [5, Vol. II, p. 171]. (There, $\varphi(t) = 1/|\log t|$.) One also has to use the fact

$$\varphi(ct) < k\varphi(t), \qquad c > 1, \qquad 0 < t < 1/4C, \tag{2.2.1}$$

which is an easy consequence of the particular form that $\varphi(t)$ has.

3. Final steps. Consider

$$\varphi(t) = \left[\int_t^1 \bar{w}(s)\,\frac{ds}{s}\right]^{-1}, \qquad 0 < t < 1/2,$$

where $\bar{w}(s) = \max(|\log s|^{-\delta}, w(s))$, $0 < \delta < \frac{1}{4}$, $0 < s < \frac{1}{2}$ and $w(s)$ as in (1.2). On account of the particular form of $\varphi(t)$, we get

$$[\varphi(t)]^{-1} = o(\log(1/t)), \qquad t \to 0. \tag{3.1}$$

Let $f(x)$ be the function whose existence is assured by Theorem 2.1 for the particular choice we made of $\varphi(t)$. Let $\varepsilon > 0$ be a positive constant, F and C be the perfect subset and the constant of Lemma 2.2. Call $\bar{f}$ any continuous extension of f from F to $[0, 2\pi]$ such that

$$|\bar{f}(x_1) - \bar{f}(x_2)| < C\varphi(|x_1 - x_2|),$$
$$x_1, x_2 \in [0, 2\pi] \text{ and } 0 < |x_1 - x_2| < 1/2. \tag{3.2}$$

Write now $f = \bar{f} + g$ and consider

$$\int_F\left(\int_0^{2\pi} |g(x) - g(y)|\,\frac{\bar{w}(|x-y|)}{|x-y|}\,dy\right)dx. \tag{3.3}$$

Since $g(x) = 0$, $x \in F$, it is enough to estimate

$$\int_F\left(\int_G |g(y)|\,\frac{\bar{w}(|x-y|)}{|x-y|}\,dy\right)dx. \tag{3.4}$$

Here, G denotes the complement of F with respect to $[0, 2\pi]$. Write $G = \bigcup_1^\infty I_k$, $\mathring{I}_i \cap \mathring{I}_j = \varnothing$, $i \neq j$, where $\operatorname{dist}(I_k, F)$ satisfies

$$|I_k| < \operatorname{dist}(I_k, F) < 2|I_k|, \quad k = 1, 2, \ldots. \tag{3.5}$$

On account of the definition of $\bar{f}$ and the above inequality we have

$$\int_{I_k} |g(y)|\, dy < K_0 \varphi(|I_k|)|I_k|, \tag{3.6}$$

which is valid for all k and a suitable choice of K_0.

Interchanging the order of integration in (3.4) and using (3.6) we get

$$\sum_1^\infty \int_{I_k} |g(y)| \left(\int_F \frac{\bar{w}(|x-y|)}{|x-y|}\, dx \right) dy \leqslant (\text{constant}) \sum_1^\infty \left(\int_{I_k} |g(y)|\, dy \right) [\varphi(|I_k|)]^{-1}$$

$$\leqslant \text{constant} \sum_1^\infty |I_k|. \tag{3.7}$$

The finiteness of (3.3) shows that

$$\int_{-\delta}^{\delta} |g(x) - g(x+t)| \frac{\bar{w}(|t|)}{|t|}\, dt < \infty \tag{3.8}$$

for almost every x in F. By construction the Fourier series of f diverges a.e. The Fourier series of $\bar{f}$ converges a.e. because

$$\int_0^{2\pi} \int_0^{2\pi} |\bar{f}(x) - \bar{f}(y)|^2 \frac{1}{|x-y|}\, dx\, dy < \infty. \tag{3.9}$$

(By construction we have that $\varphi(t) \leqslant C(1/|\log t|^{1-\delta})$ with $0 < \delta < 1/4$.) (See Theorem 1.14 in [5, p. 164].)

Finally, (3.8) above also holds for $w(t)$ on account of the fact that $w(t) \leqslant \bar{w}(t)$. This concludes the proof.

4. A few additional remarks. The Dini condition (1.1) for $x \in E$, $|E| > 0$, is not comparable to the condition

$$\frac{1}{h} \int_0^h |f(x+t) - f(x)|\, dt = O\left(\frac{1}{|\log h|} \right), \qquad x \in E. \tag{4.1}$$

The reason for that is the fact that (1.1) assures convergence of $S_m(f)$ everywhere in E, while (4.1) assures only convergence a.e. in E (see [4] and [5, Vol. I, p. 302 (2.1)]).

Theorem 2.1 above was proved by Marcinkiewicz to show that condition (4.1) can not be replaced by any similar weaker one.

Finally, I would like to thank Professor Zygmund for many opportune and helpful suggestions concerning the matters of this paper.

References

1. L. Carleson, *On the convergence and growth of partial sums of Fourier Series*, Acta Math. **116** (1966), 135–157.

2. Richard Hunt, *On the convergence of Fourier Series*, Orthogonal Expansions and their Continuous Analogues, Southern Illinois University Press, 1968, pp. 235–255.

3. J. Marcinkiewicz, *Sur les séries de Fourier*, Fund. Math. **27** (1936), 38–69.

4. _____, *On the convergence of Fourier Series*, J. London Math. Soc. **10** (1935), 264–268.

5. A. Zygmund, *Trigonometric series*, Vols. I and II, Cambridge University Press, 1968.

Department of Mathematics, University of Illinois – Chicago Circle, Chicago, Illinois 60680