FIXED POINTS AND BOUNDARIES

ERIC CHANDLER

ABSTRACT. A lemma of Ludvik Janos is used to show that if a nonexpansive self-map \( T \) of a compact set \( X \) is contractive on \( \Delta'X \), the boundary of \( X \) in \( \text{co} \ X \), then \( T \) has a fixed point in \( X \). It is further proven that if \( T(\Delta'X) \cap \Delta'X = \emptyset \), or if \( T \) maps any point \( y \) of \( X \) away from \( \Delta'X \), then \( T \) has a fixed point in \( X \).

In the results which follow, \( X \) is a compact subset of a strictly convex normed linear space \( E \) and \( T : X \rightarrow X \) is nonexpansive. For a subset \( S \) of \( E \) we shall let \( \Delta' S \) denote the boundary of \( S \) in \( \text{co} \ S \), the closed convex hull of \( S \).

In [1, Lemma 1] we stated the following version of a result of Janos [4, Lemma 3.1]:

**Lemma 1.** There exists a nonexpansive retraction \( r \) of \( X \) onto the compact set \( C_T = \bigcap_{i=1}^{\infty} T^i X \). Furthermore, this \( r \) is in the closure, in the pointwise topology of \( X^X \), of the set \( \{ T^i \}_{i=1}^{\infty} \).

**Lemma 2.** \( \Delta' C_T \subseteq \Delta' X \).

**Proof.** We claim that \( (\text{co} \ C_T) \cap X = C_T \), for if not, there exists an \( x \in X \setminus C_T \) such that \( x = a_1 x_1 + \cdots + a_n x_n \) where \( x_i \in C_T, a_i > 0 \), and \( \sum_{i=1}^{n} a_i = 1 \). Edelstein has shown [2, Proposition 2] that if \( f : E \rightarrow E \) is a nonexpansive mapping of a strictly convex normed linear space \( E \) into itself and if \( f|_A, A \subset E \), is an isometry, then

\[
x = ax_1 + (1 - a)x_2 \quad \text{for } x_1, x_2 \in A \text{ and } 0 < a < 1
\]

implies that

\[
f(x) = af(x_1) + (1 - a)f(x_2).
\]

Thus, since \( r|_{C_T} \) is the identity map we have \( rx = r(a_1 x_1 + \cdots + a_n x_n) = a_1 rx_1 + \cdots + a_n rx_n = a_1 x_1 + \cdots + a_n x_n = x \). But \( rx \in C_T \) and so \( x \in C_T \) which is impossible. Thus \( (\text{co} \ C_T) \cap X = C_T \).

Now if \( y \in \Delta' C_T \) and \( U \) is any open set containing \( y \), then the open set \( U \setminus C_T \) contains a point in \( (\text{co} \ C_T) \setminus C_T \), and so also a point \( z \) in \( (\text{co} \ C_T) \setminus C_T \). But \( z \) cannot belong to \( X \) since \( (\text{co} \ C_T) \cap X = C_T \). Thus \( z \in (\text{co} X) \setminus X \) and so \( y \in \Delta' X \). Hence \( \Delta' C_T \subseteq \Delta' X \).

**Theorem 1.** If \( T|_{\Delta'X} \) is contractive (i.e. if \( \| Tx - Ty \| < \| x - y \| \) for all \( x, y \) in \( \Delta' X \)) then \( T \) has a fixed point in \( X \).

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Proof. $T$ maps $C_T$ onto $C_T$ and $C_T$ is compact. If $C_T$ is convex, then Schauder's theorem implies that $T$ has a fixed point in $C_T$. If $C_T$ is not convex, then $\Delta'C_T \neq \emptyset$ and obviously $|\Delta'C_T| > 2$. By Lemma 2, $\Delta'C_T \subseteq \Delta'X$. Any nonexpansive mapping of a compact metric space onto itself is an isometry [3, Satz 1b] and so $T$ is an isometry on $C_T$ and thus on $\Delta'C_T$. This is a contradiction, and so $C_T$ must be convex and thus contain a fixed point of $T$.

**Theorem 2.** If $T(\Delta'X) \cap \Delta'X = \emptyset$ then $T$ has a fixed point in $X$.

Proof. As before, if $C_T$ is convex, we are done. So suppose $\Delta'C_T \neq \emptyset$.

Now the retraction $r$ is the identity map $I$ on $C_T$, and so $I$ is in the closure, in the pointwise topology, of the set $(T^t)_{t=1}^\infty$. This means that for any finite subset of $C_T$, in particular for any $x, y \in \Delta'C_T$, we can find a subsequence $(n_i)$ so that $T^{n_i}x \to Ix = x$ and $T^{n_i}y \to Iy = y$.

Let us assume then that $x, y \in \Delta'C_T$ and, without loss of generality, that the open line segment $L = \{ax + (1-a)y | 0 < a < 1\}$ lies entirely in the complement of $C_T$. Since $Tx$ and $Ty$ do not belong to $\Delta'C_T$ (for by Lemma 2, $\Delta'C_T \subseteq \Delta'X$) there exists an $a$, $0 < a < 1$, such that $z = ax + (1-a)Ty$ belongs to $C_T$. Let $(n_i)$ be the subsequence associated with $x, y$ such that $T^{n_i}x \to x$ and $T^{n_i}y \to y$.

Then by [2, Proposition 2], $T^{n_i-1}z = T^{n_i-1}(ax + (1-a)Ty) = aT^{n_i}x + (1-a)T^{n_i}y$ which approaches the point $ax + (1-a)y$ on $L$ in the complement of $C_T$. This is impossible since $T: C_T \to C_T$. Thus $C_T$ is convex and so contains a fixed point of $T$.

(Note that in the proof of this theorem we could have assumed as the hypothesis that if there exists an $n > 1$ such that $T^n\Delta'X \cap \Delta'X = \emptyset$ then $T$ has a fixed point in $X$.)

Given a subset $Q \subseteq X$ and a point $y \in X$ we say that “$T$ maps $y$ away from $Q$” if $||Ty - x|| > ||y - x||$ for all $x \in Q$.

**Theorem 3.** If there exists a point $y \in X$ such that $T$ maps $y$ away from $\Delta'X$, then $T$ has a fixed point in $X$.

Proof. The real-valued map $\phi(x) = ||Ty - x|| - ||y - x|| > 0$ on the compact set $\Delta'X$ is continuous and so there exists an $\epsilon > 0$ such that $||Ty - x|| - ||y - x|| > \epsilon$ for all $x \in \Delta'X$. If $\Delta'C_T \neq \emptyset$, there is a point $z$ in the compact set $\Delta'C_T$ such that $r = ||y - z|| = \text{dist}(y, \Delta'C_T)$. Consider the open sphere $S = S(y, r + \epsilon)$, and let $w \in S \cap (\text{co} C_T \setminus C_T)$. On the closed line segment $[wz]$ there exists a point $z_0 \in \Delta'C_T$ such that the open line segment $(wz_0)$ lies entirely in $\text{co} C_T \setminus C_T$.

Obviously $Tz_0 \in C_T \cap S(Ty, r + \epsilon)$. We claim that $S(Ty, r + \epsilon) \cap \Delta'C_T = \emptyset$, for if $x \in \Delta'C_T$ then $||Ty - x|| > r$ and $||Ty - x|| - ||y - x|| > \epsilon$ so that $||Ty - y|| > r + \epsilon$. Thus $Tz_0 \in C_T$ but $Tz_0 \not\in \Delta'C_T$.

Let us suppose that the $w$ above is represented: $w = a_1x_1 + \cdots + a_kx_k$ where $x_i \in C_T$, $\Sigma_{i=1}^k a_i = 1$ and $a_i > 0$. The points $Tx_i$ all lie in $C_T$ and thus the point $w_0 = a_1Tx_1 + \cdots + a_kTx_k$ is in $\text{co} C_T$. Now on the open line segment $(w_0Tz_0)$ there must exist a point $p \in C_T$, for otherwise $Tz_0 \in \Delta'C_T$ which is impossible. Let $p = aTz_0 + (1-a)w_0$ where $0 < a < 1$. Then $p = aTz_0 + \Sigma_{i=1}^k(1-a)a_iTx_i$ and $a + \Sigma_{i=1}^k(1-a)a_i = 1$. 


For the points \( z_0, x_1, \ldots, x_k \) let \( \{n_j\} \) be a subsequence of \( \{n\} \) for which 
\( T^{n_j} z_0 \to z_0 \) and \( T^{n_j} x_i \to x_i \), \( 1 \leq i \leq k \). (We can do this since the identity on \( C_T \) is in the pointwise closure of \( \{T^n\}_{n=1}^\infty \).) Then by [2, Proposition 2],
\[
T^{n_j-1}p = T^{n_j-1}\left[ aTz_0 + \sum_{i=1}^{k} (1 - a)a_iTx_i \right]
\]
which approaches the point \( az_0 + (1 - a)\sum_{n=1}^{k} a_i x_i = az_0 + (1 - a)w \). Thus 
\( T^{n_j-1}p \) approaches a point lying on the open line segment \( (wz_0) \). This is not possible since \( (wz_0) \) lies entirely outside \( C_T \) and \( T: C_T \to C_T \). Thus \( \Delta' C_T = \emptyset, C_T \) is convex, and so \( T \) has a fixed point in \( C_T \).

References


Department of Mathematics, Randolph-Macon Woman’s College, Lynchburg, Virginia 25403