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ON THE COMPACTIFICATION OF STRONGLY PSEUDOCONVEX SURFACES

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ABSTRACT. In this paper, we shall prove that the compactification of a strongly pseudoconvex surface is either a projective algebraic or an Inoue surface.

Furthermore, we shall construct an example of a strongly pseudoconvex surface X which admits two distinct compactifications: One M' projective algebraic and the other one M (highly) nonalgebraic.

Throughout this paper, **C**-analytic surfaces (compact or noncompact) will mean 2-dimensional **C**-analytic manifolds. Purely 1-dimensional **C**-analytic spaces will be referred to simply as analytic curves.

1. The main problem.

DEFINITION 1 [1]. A **C**-analytic surface X is called *strongly pseudoconvex* if there exist

(i) a compact analytic curve $E \subset X$,

(ii) an exhaustion function $\phi \in C^{\infty}_{\mathbf{R}}(X)$ such that the Levi form

$$L(\phi)_x := \sum \frac{\partial^2 \phi(x)}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

is positive definite for any $x \in X \setminus E$, with $1 \leqslant i, j \leqslant 2$.

REMARK. It can be proved that E is actually *exceptional* in the sense of Grauert [1].

DEFINITION 2 [4]. Let M be a **C**-analytic surface and let $E \subset X$ be a compact analytic curve.

(a) E is called an *exceptional curve of the first kind* if $E \xrightarrow{\sim} \mathbf{P}_1$ and $E^2 = -1$.

(b) The **C**-analytic surface M is said to be *minimal* if M is free from exceptional curves of the first kind.

From now on, all **C**-analytic surfaces (compact or noncompact) are assumed to be minimal.

Let M be a compact **C**-analytic surface and let $a(M) :=$ the transcendental degree of the field of meromorphic functions on M. It is well known that $0 \leqslant a(M) \leqslant 2$. Following Kodaira [4, Vol. III], compact **C**-analytic surfaces can be classified as follows (see also [3]):

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THEOREM 1. *Let M be a compact* **C**-*analytic surface. Then*

(i) *M is projective algebraic iff $a(M) = 2$.*

(ii) *M is elliptic iff $a(M) = 1$.*

If $a(M) = 0$, then M must be one of the following types:

(iii) *M is a 2-dimensional torus.*

(iv) *M is a K_3 surface i.e. $c_1(M) = b_1(M) = 0$.*

(v) *M is a nonelliptic Hopf surface i.e. $b_2(M) = 0$, $b_1(M) = 1$ and M contains at least one analytic compact curve.*

(vi) *$b_2(M) = 0$, $b_1(M) = 1$ and M contains no analytic curves.*

(vii) *$b_2(M) > b_1(M) = 1$.*

REMARKS. (a) Here $c_1(M)$ (resp. $b_1(M)$, resp. $b_2(M)$) denote the first Chern class (resp. the first Betti number, resp. the second Betti number) of M.

(b) Although a complete classification of compact **C**-analytic surfaces of types (vi) and (vii) is not known yet, recently Inoue has constructed explicit examples of such surfaces [3].

Our main purpose here is to investigate the following

Problem A. Let M be a compact **C**-analytic surface and let us assume that there exists a compact analytic curve $\Gamma \subset M$ such that $X := M \setminus \Gamma$ is strongly pseudoconvex. What are the analytic structures M might be equipped with?

In the special case where X is assumed to be Stein, this problem has been settled by Howard [2]. Our approach here is influenced by the technique in [2] and by recent construction of new compact **C**-analytic surfaces without meromorphic functions [3].

REMARK. Since X is strongly pseudoconvex, hence holomorphically convex, one can easily check that Γ is connected. Let $\Gamma = \bigcup_i \Gamma_i$ where Γ_i are the irreducible compact analytic curves.

2. The main theorem. We are now in a position to provide a complete answer to Problem A.

THEOREM B. *Let M be a compact* **C**-*analytic surface and assume that there exists a compact analytic curve $\Gamma \subset M$ such that $M \setminus \Gamma =: X$ is strongly pseudoconvex admitting some exceptional curve E. Then either* (i) *M is projective algebraic or* (ii) *$b_2(M) > b_1(M) = 1$.*

First of all we shall need the following result:

LEMMA 2. *Let M, Γ, X and E be as in Theorem* B. *Then the natural map*

$$r: H^1(M, \mathbf{C}) \to H^1(\Gamma, \mathbf{C})$$

is injective.

PROOF. In fact one has the following exact sequence of cohomology groups with compact support:

$$\cdots \to H_c^1(M \setminus \Gamma, \mathbf{C}) \to H^1(M, \mathbf{C}) \to H^1(\Gamma, \mathbf{C}) \to .$$

Since M is nonsingular of $\mathbf{R}$-dimension $= 4$, Poincaré duality tells us that

$$H_c^1(M \setminus \Gamma, \mathbf{C}) \simeq H_3(M \setminus \Gamma, \mathbf{C}) = H_3(X, \mathbf{C}).$$

Since X is strongly pseudoconvex, it follows from **[5]** that

$$H_3(X, \mathbf{C}) \simeq H_3(E, \mathbf{C}) = 0.$$

Hence it follows that r is injective. Q.E.D.

PROOF OF THEOREM B. *Step* 1. Let us assume that $a(M) = 1$.

Following Kodaira **[4]** there exist a compact analytic curve Ξ and an analytic map π: $M \to \Xi$ such that generically the fibres of π are elliptic curves. Since $a(M) = 1$, for any compact connected analytic curve $C \subset M$, $\pi(C)$ must be a point in Ξ (see **[4**, Theorem 4.3, p. 1184**]**). Now since E and Γ are compact analytic curves in M, there exists a finite set $T \subset \Xi$ such that $\pi(E) \cup \pi(\Gamma) \subset T$. Hence for any $z \in \Xi \setminus T$, the elliptic curve $\Delta := \bar{\pi}^1(z) \subset M \setminus (E \cup \Gamma) \simeq X \setminus E$. But the existence of the compact analytic subvariety Δ of positive dimension contradicts the strong plurisubharmonicity of ϕ on $X \setminus E$ (see Definition 1(ii)).

Step 2. Let us assume that $a(M) = 0$.

(a) If M is a 2-dimensional torus, an argument in **[2]** shows that Γ must be a nonsingular elliptic curve. Consequently, $b_1(M) = 4$ and $b_1(\Gamma) = 2$. But this will contradict the injectivity of the map r in Lemma 2.

(b) If M is a K_3 surface, let L be the line bundle determined by the divisor $\Gamma = \bigcup \Gamma_i$ and let $a_{ij} := \Gamma_i \cdot \Gamma_j$.

In view of the hypothesis that $a(M) = 0$, an argument in **[2]** and **[4]** shows that the matrix (a_{ij}) must be negative definite.

Claim. M cannot be the compactification of any strongly pseudoconvex surface.

In fact if it were, in view of the negative definiteness of (a_{ij}), a result in **[3]** tells us that there exist a 2-dimensional normal compact $\mathbf{C}$-analytic space $\hat{M}$, a point $\{*\} \subset \hat{M}$ and a surjective holomorphic map θ: $M \to \hat{M}$ inducing a biholomorphism $M \setminus \Gamma \simeq \hat{M} \setminus \{*\}$. Since $\hat{M} \setminus \{*\} \simeq X$ is holomorphically convex, in view of Riemann's extension theorem, any global holomorphic function on $\hat{M} \setminus \{*\}$ can be extended to $\hat{M}$. This implies the existence of nonconstant global holomorphic functions on the compact analytic space $\hat{M}$. Contradiction!

Step 3. Let us consider the following natural map

$$\alpha\colon H^2(M, \mathbf{C}) \to H^2(E, \mathbf{C})$$
$$c_1(E) \mapsto E \cdot E.$$

Since E is exceptional, $E^2 < 0$. Therefore α is not a zero map. Consequently $H^2(M, \mathbf{C})$ and hence $b_2(M) \neq 0$.

In view of Theorem 1 above, M is either projective algebraic or $b_2(M) > b_1(M) = 1$. Q.E.D.

3. The main example. We are now in a position to exhibit an example showing that both alternatives (i) and (ii) in Theorem B indeed occur (and even simultaneously).

EXAMPLE C. In [3] a compact **C**-analytic surface M is explicitly constructed with $b_1(M) = 1$. Furthermore, M contains exactly two compact analytic curves:

(i) a rational curve Γ with an ordinary double point with $\Gamma^2 = 0$,

(ii) an elliptic curve E with $E^2 = -1$.

Also it is proved that $X := M \setminus \Gamma$ is biholomorphic to a holomorphic line bundle over E with E as its zero section. In view of (ii) a result in [1] tells us that X is actually a strongly pseudoconvex manifold admitting E as its exceptional curve. Moreover, since $K = [-\Gamma - E]$ where K is the canonical line bundle on M, one has, in view of (i) and (ii), $b_2(M) = -K^2 = 1$.

On the other hand since X is biholomorphic to a line bundle, the fibres of the latter can be projectivized so that X can be realized as a Zariski open subset of some ruled surface, say M' over E.

Consequently, we obtain here an example of a strongly pseudoconvex surface which admits 2 distinct compactifications: One projective algebraic M' and the other one M, (highly) nonalgebraic.

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