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TWO-DIMENSIONAL, ALMOST PERIODIC LINEAR SYSTEMS WITH PROXIMAL AND RECURRENT BEHAVIOR

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ABSTRACT. We prove that there exist two-dimensional, almost periodic linear systems, with arbitrary basis of frequencies, the angular coordinates of whose solutions have both proximal and recurrent behavior. Such behavior is completely unlike that of any periodic system.

1. Introduction. Consider a two-dimensional, linear differential equation

$$\dot{x} = A(t)x, \quad \operatorname{trace} A(t) \equiv 0, \tag{$*$}$$

where $A(t)$ is periodic. If $x_1(t)$ and $x_2(t)$ are solutions of $(*)$, let $\Theta(x_1(t), x_2(t))$ denote the angular separation between them. Say x_1 and x_2 are *proximal* if $\inf_t \Theta(x_1(t), x_2(t)) = 0$. If every pair $x_1(t)$, $x_2(t)$ of solutions of $(*)$ is proximal, then a periodic change of variables $x = p(t)y$ takes $(*)$ to the form

$$\dot{y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y. \tag{$**$}$$

It follows that $(*)$ has a periodic solution $x_0(t)$ such that $\lim_{|t|\to\infty} \Theta(x(t), x_0(t)) = 0$ for every solution $x(t)$.

Now suppose $A(t)$ is *almost* periodic. Suppose $\inf_t \Theta(x_1(t), x_2(t)) = 0$ for each pair of solutions $x_1(t)$, $x_2(t)$ of $(*)$. We look for analogies with the periodic case. For example, let $(\Sigma, \mathbf{R})$ be the projective flow (2.3) generated by $(*)$. We ask if $(\Sigma, \mathbf{R})$ admits a minimal, almost automorphic subflow $(M, \mathbf{R})$ (2.1; see **[13]**). Such an M would be an analogue of the periodic solution $x_0(t)$ mentioned in the last paragraph.

We will show that the answer to this question is "no". We will show that there are equations $(*)$ whose projective flow is a minimal, *proximal* extension of the hull **[11]** of A (see 2.1). This means that:

(i) $\inf_t \Theta(x_1(t), x_2(t)) = 0$ for all solutions $x_1(t)$, $x_2(t)$ of $(*)$;

(ii) the angular coordinate $\theta(t)$ of every solution $x(t)$ of $(*)$ wanders densely through the set of all directions in the sense that, if $\theta_0 \in [0, 2\pi)$, then there is a sequence $t_n \to \infty$ such that $A(t + t_n) \to A(t)$ uniformly, and $\theta(t_n) \to \theta_0 \bmod 2\pi$. We will also find equations $(*)$ such that, in addition to the above, $(\Sigma, \mathbf{R})$ is strictly ergodic (2.1).

We will use ideas similar to those of Glasner and Weiss **[6]**. They in turn refer to a paper of Anosov and Katov **[1]** and Fathi-Herman **[4]**. We note that Ellis (see **[3]**)

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has developed a technique for producing examples like ours, and higher-dimensional analogues. Our development is relatively concrete; also, our proofs are self-contained, and involve only the most simple constructions.

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2. Preliminaries.

2.1 DEFINITIONS. Let X be a compact metric space. A (real) *flow* on X, denoted $(X, \mathbf{R})$, is given by a continuous map $\Phi: X \times \mathbf{R} \to X: (x, t) \to x \cdot t$ such that: (i) $x \cdot 0 = x$ $(x \in X)$; (ii) $x \cdot (t + s) = (x \cdot t) \cdot s$ $(x \in X; t, s \in \mathbf{R})$. Say $(X, \mathbf{R})$ is *minimal* if every orbit $\{x \cdot t | t \in \mathbf{R}\}$ is dense in X $(x \in X)$. Say $(X, \mathbf{R})$ is *uniquely ergodic* if there is a unique $\mathbf{R}$-invariant measure **[10]** on X; if, in addition, $(X, \mathbf{R})$ is minimal, then it is *strictly ergodic*. Say $(X, \mathbf{R})$ is *almost periodic* (a.p) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(x \cdot t, y \cdot t) < \varepsilon$ $(x \in X, y \in X, t \in \mathbf{R})$. Here d is some metric on X. If $(Y, \mathbf{R})$ is another flow with Y compact metric, then $(X, \mathbf{R})$ is an *extension* of $(Y, \mathbf{R})$ if there is a continuous surjection $\pi: X \to Y$ such that $\pi(x \cdot t) = \pi(x) \cdot t$ $(x \in X, t \in \mathbf{R})$. In this situation, suppose $\inf_t d(x_1 \cdot t, x_2 \cdot t) = 0$ whenever $\pi(x_1) = \pi(x_2)$; then $(X, \mathbf{R})$ is a *proximal extension* of $(Y, \mathbf{R})$. Also, if $(Y, \mathbf{R})$ is a.p. and minimal, and if $\operatorname{card} \pi^{-1}(y) = 1$ for some $y \in Y$, then $(X, \mathbf{R})$ is *almost automorphic* **[13]**.

2.2 DEFINITIONS. Let Ω be a compact metric space, and let $(\Omega, \mathbf{R})$ be an a.p. minimal flow. Let $L(2)$ be the set of 2×2, real matrices, with

$$|B| = \sup\{|Bx|: |x| = 1\};$$

here $|\cdot|$ is the Euclidean norm on $\mathbf{R}^2$. Let $a: \Omega \to L(2)$ be continuous, and consider the linear differential equations

$$\dot{x} = a(\omega \cdot t)x \qquad (x \in \mathbf{R}^2, \omega \in \Omega). \tag*{$(1)_{\omega,a}$}$$

Every almost periodic ODE $\dot{x} = A(t)x$ induces an a.p. minimal flow $(\Omega, \mathbf{R})$ (Ω = hull of A), and a collection of equations $(1)_{\omega,a}$ **[11]**. From now on, we fix some a.p. minimal flow $(\Omega, \mathbf{R})$.

2.3 DEFINITIONS. Equations $(1)_{\omega,a}$ define a flow on $\Omega \times \mathbf{R}^2$ via $(\omega, x_0) \cdot t = (\omega \cdot t, x(t))$, where $x(t)$ is the solution to $(1)_{\omega,a}$ such that $x(0) = x_0$ **[11]**. Let $\mathbf{P}^1$ be real, one-dimensional projective space, and let $\Sigma = \Omega \times \mathbf{P}^1$. The flow $(\Omega \times \mathbf{R}^2, \mathbf{R})$ induces a flow on Σ via $(\omega, \mathfrak{l}) \cdot t = (\omega \cdot t, \mathfrak{l}(t))$. Here $\mathfrak{l}$ is a line through the origin in $\mathbf{R}^2$, and $\mathfrak{l}(t)$ its image after time t. Denote this flow by $(\Sigma_a, \mathbf{R})$ to indicate the dependence on a. Then $(\Sigma_a, \mathbf{R})$ describes the angular evolution of solutions of equations $(1)_{\omega,a}$. It is the *projective flow* defined by equations $(1)_{\omega,a}$.

2.4 DEFINITION. Since $(\Omega, \mathbf{R})$ is a.p. minimal, it may be given the structure of a compact, abelian, topological group with dense subgroup $\mathbf{R}$ **[2]**. Let ν be normalized Haar measure on Ω. Define

$$C_0(\Omega) = \left\{ b: \Omega \to \mathbf{R} \,\Big|\, \int_\Omega b(\omega)\, d\nu(\omega) = 0 \right\}.$$

2.5 REMARKS. Let $b \in C_0(\omega)$, and consider the equations

$$\dot{x} = \begin{pmatrix} 0 & -b(\omega \cdot t) \\ b(\omega \cdot t) & 0 \end{pmatrix} x \qquad (\omega \in \Omega,\ t \in \mathbf{R}). \tag{$(2)_{\omega,b}$}$$

It is known that the flow on Σ induced by equations $(2)_{\omega,b}$ is minimal if (and only if) $\int_0^t b(\omega \cdot s)\, ds$ is unbounded for some (hence any) $\omega \in \Omega$. It is strictly ergodic, if, in addition, the equation $r(\omega \cdot t) - r(\omega) = \int_0^t b(\omega \cdot s)\, ds$ $(\omega \in \Omega, t \in \mathbf{R})$ has no ν-measurable solution r [5], [8].

3. **Results.**

3.1 DEFINITIONS. Let $P(\Omega) = \{p: \Omega \to L(2) | p$ is continuous, $\det p(\omega) = 1$ for all $\omega \in \Omega$, and $p': \Omega \to L(2): \omega \to (d/dt)p(\omega \cdot t)|_{t=0}$ is continuous$\}$. Then, let

$$S_0 = \left\{ a: \Omega \to L(2) | a(\omega) = p^{-1}(\omega) \begin{pmatrix} 0 & -b(\omega) \\ b(\omega) & 0 \end{pmatrix} p(\omega) \right.$$
$$\left. -p^{-1}(\omega)p'(\omega) \text{ for some } p \in P(\Omega) \text{ and some } b \in C_0(\Omega) \right\}.$$

3.2 REMARK. We can and will identify S_0 with the set of all collections $(1)_{\omega,a}$ of equations that are obtained from a collection $(2)_{\omega,b}$ by a "strong Perron transformation" $x = p(\omega \cdot t)y$ $(\omega \in \Omega)$, where $\det p(\omega) \equiv 1$.

3.3 DEFINITIONS. Let $\mathcal{C} = \{a: \Omega \to L(2) | a$ is continuous$\}$. Norm $\mathcal{C}$ as follows: $\|a\| = \sup\{|a(\omega)|: \omega \in \Omega\}$; then $\mathcal{C}$ is a Banach space. Define $S = \operatorname{cls} S_0 \subset \mathcal{C}$.

3.4 REMARK. If $a \in S$, then trace $a(\omega) = 0$ $(\omega \in \Omega)$.

3.5 ASSUMPTION. From now on, assume $(\Omega, \mathbf{R})$ is not periodic (i.e., Ω is not a circle). Recall $(\Omega, \mathbf{R})$ is a.p. minimal (2.2).

3.6 PROPOSITION. *There is a residual subset of S_1 of S such that, if $a \in S_1$, then $(\Sigma_a, \mathbf{R})$ is minimal.*

PROOF. Let $\{O_n: n \geq 1\}$ be a countable base for the topology of Σ. For each $n \geq 1$, let $C_n = \{a \in S|$ the flow $(\Sigma_a, \mathbf{R})$ contains a minimal set which does not intersect $O_n\}$. Then C_n is closed in S. For, let $a_m \to a$, $a_m \in C_n$. Then $(\Sigma_{a_m}, \mathbf{R})$ contains a minimal set F_m such that $F_m \cap O_n = \varnothing$ $(m \geq 1)$. Perhaps choosing a subsequence, we may assume $F_m \to F \subset \Sigma$ in the Hausdorff metric on the space of closed subsets of Σ [13]. Then F is a compact invariant subset of $(\Sigma_a, \mathbf{R})$, and $F \cap O_n = \varnothing$. Since F contains a minimal subset, C_n is closed.

Now, $\{b \in C_0(\Omega) | \int_0^t b(\omega \cdot s)\, ds$ is unbounded for some $\omega \in \Omega\}$ is dense in $C_0(\Omega)$, since $(\Omega, \mathbf{R})$ is not periodic. By 2.5 and the definition of S, $\{a \in S | (\Sigma_a, \mathbf{R})$ is minimal$\}$ is dense in S. Hence S_n is nowhere dense in S, so $S_1 = S \sim \bigcup_{n=1}^{\infty} S_n$ is a residual subset of S. Clearly $a \in S_1 \Leftrightarrow (\Sigma_a, \mathbf{R})$ is minimal.

3.7 PROPOSITION. *There is a residual subset S_2 of S such that, if $a \in S_2$, then equation $(1)_{\omega,a}$ admits an unbounded solution for some (hence all) $\omega \in \Omega$.*

PROOF. For each integer $N \geq 1$, let $C_N = \{a \in S|$ every solution $x(t)$ to $(1)_{\omega,a}$ such that $|x(0)| = 1$ satisfies $|x(t)| \leq N$ $(\omega \in \Omega, t \in \mathbf{R})\}$. Clearly C_N is a closed subset of S. Define $S_2 = S \sim \bigcup_{N=1}^{\infty} C_N$; then $S_2 = \{a \in S \mid$ some equation $(1)_{\omega,a}$ admits an unbounded solution$\}$.

To prove 3.7, we must show that each C_N has dense complement in S. Using the definition of S and the fact that $\det p(\omega) = 1$ $(\omega \in \Omega)$ for all $p \in P(\Omega)$, we see that it suffices to prove the following. Let $b \in C_0(\Omega)$, $N > 1$, and $\varepsilon > 0$ be given. Then there is a $t_0 = t_0(N, b, \varepsilon)$ in $\mathbf{R}$ and an $\omega_0 = \omega_0(b)$ in Ω such that, for each $\theta \in \mathbf{R}$, the ε-neighborhood of $\left(\begin{smallmatrix} 0 & -b \\ b & 0 \end{smallmatrix}\right)$ in S contains an element $a = a(\theta)$ such that equation $(1)_{\omega_0,a}$ admits a solution $x(t)$ with $x(t_0)/|x(t_0)| = (\cos\theta, \sin\theta)$, and $|x(t_0)|/|x(0)| > N$.

So, choose a function $b_1 \in C_0(\Omega)$ such that $\sup\{|b(\omega) - b_1(\omega)| \colon \omega \in \Omega\} < \varepsilon/2$, and $\int_0^t b_1(\omega \cdot s)\, ds$ is bounded for some (hence all) $\omega \in \Omega$. We may write

$$\int_0^t b_1(\omega \cdot s)\, ds = B_1(\omega \cdot t) - B_1(\omega),$$

where $B_1 \in C_0(\Omega)$. Let

$$p_1(\omega) = \begin{pmatrix} \cos B_1(\omega) & -\sin B_1(\omega) \\ \sin B_1(\omega) & \cos B_1(\omega) \end{pmatrix} \qquad (\omega \in \Omega).$$

Note that the change of variables $x = p_1(\omega \cdot t)y$ takes equations $(2)_{\omega,b_1}$ to the form $\dot{y} = 0$ for all $\omega \in \Omega$. Pick $\omega_0 \in \Omega$ such that $B_1(\omega_0) = 0$.

Next, choose a function $u \in C_0(\Omega)$ such that:

(i) $\sup\{\exp u(\omega) \mid \omega \in \Omega\} > N$;

(ii) if $u'(\omega) = (d/dt)u(\omega \cdot t)|_{t=0}$, then $|u'(\omega)| < \varepsilon/2$ $(\omega \in \Omega)$;

(iii) $u(\omega_0) = 0$.

We will see momentarily that such a u may be found. Let

$$p_2(\omega) = \begin{pmatrix} \exp -u(\omega) & 0 \\ 0 & \exp u(\omega) \end{pmatrix} \qquad (\omega \in \Omega);$$

also let

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \quad \text{for any } \varphi \in \mathbf{R}.$$

Choose t_0 so that $\exp u(\omega_0 \cdot t_0) > N$, and let

$$p_1(\omega_0 \cdot t_0) = \begin{pmatrix} \cos\bar{\theta} & -\sin\bar{\theta} \\ \sin\bar{\theta} & \cos\bar{\theta} \end{pmatrix}.$$

It may now be checked that the change of variables

$$x = p_1(\omega \cdot t)p_2(\omega \cdot t)R_{-\theta+\bar{\theta}}p_1^{-1}(\omega \cdot t)y$$

takes equations $(2)_{\omega,b_1}$ to the form $\dot{y} = Q(\omega \cdot t)y$, where

$$Q(\omega) = p_1(\omega)R_{\theta-\bar{\theta}}p_2^{-1}(\omega)p_2'(\omega)R_{-\theta+\bar{\theta}}p_1^{-1}(\omega) - p_1(\omega)(p_1^{-1})'(\omega) \qquad (\omega \in \Omega). \quad (3)$$

Since

$$\begin{pmatrix} 0 & -b_1(\omega) \\ b_1(\omega) & 0 \end{pmatrix} = -p_1(\omega)(p_1^{-1})'(\omega),$$

and since $\|p_1 R_{\theta-\bar\theta} p_2^{-1} p_2' R_{-\theta+\bar\theta} p_1^{-1}\| < \varepsilon/2$ by choice of u, we see that

$$\left| Q(\omega) - \begin{pmatrix} 0 & -b(\omega) \\ b(\omega) & 0 \end{pmatrix} \right| < \varepsilon \qquad (\omega \in \Omega).$$

Also, if $y(t)$ satisfies $\dot y = Q(\omega \cdot t)y$, and $y(0) = (\cos(\theta - \bar\theta), \sin(\theta - \bar\theta))$, then $|y(t_0)| > N$, and $y(t_0)/|y(t_0)| = (\cos\theta, \sin\theta)$.

Finally, we show that u may be chosen in the indicated way. We proceed as in [7, §4]. Corresponding to $(\Omega, \mathbf{R})$ is some module of frequencies $\{\lambda_K\}_{K=1}^{\infty}$. Any Bohr almost periodic function f: $\mathbf{R} \to \mathbf{R}$ whose frequency module is contained in $\{\lambda_K\}$ may be viewed as a continuous function on Ω, as follows. Let $\omega_0 \in \Omega$ be as above, and define $\tilde f(\omega_0 \cdot t) = f(t)$ $(t \in \mathbf{R})$; then $\tilde f$ admits a unique continuous extension to Ω. Now let $\tilde u(\omega_0 \cdot t) = [\ln(2N)] \cdot \sin \lambda_I t$, where $\lambda_I \in \{\lambda_K\}$ is chosen so that $0 \neq |\lambda_I| < (\varepsilon/2)\ln(2N)$. This may be done because $(\Omega, \mathbf{R})$ is not periodic. Let u be the unique continuous extension of $\tilde u$ to Ω. Then $u \in C_0(\Omega)$. Also, u' is in $C_0(\Omega)$, and is the unique continuous extension to Ω of $\tilde u'(\omega_0 \cdot t) = -\lambda_I \cdot [\ln(2N)] \cdot \cos \lambda_I t$. It is clear that u satisfies the desired conditions.

3.8 Corollary. *There is a residual subset S_3 of S such that, if $a \in S_3$, then $(\Sigma_a, \mathbf{R})$ is minimal, and is a proximal extension of $(\Omega, \mathbf{R})$.*

Proof. Let $S_3 = S_1 \cap S_2$, where S_1 resp. S_2 is defined in 3.6 (resp. 3.7). If $a \in S_3$, then $(\Sigma_a, \mathbf{R})$ is minimal, and some equation $(1)_{\omega,a}$ admits an unbounded solution. If $(\Sigma_a, \mathbf{R})$ is not a proximal extension of $(\Omega, \mathbf{R})$, then [9, §7] implies that there is a minimal subflow $(M, \mathbf{R})$ of $(\Sigma_a, \mathbf{R})$ such that $\mathrm{card}(M \cap \{\omega\} \times \mathbf{P}^1) = 2$ for a residual set of $\omega \in \Omega$. Hence $M \neq \Sigma_a$. This contradicts minimality of $(\Sigma_a, \mathbf{R})$; hence $(\Sigma_a, \mathbf{R})$ is a proximal extension of $(\Omega, \mathbf{R})$.

3.9 Proposition. *Let $S_4 = \{a \in S | (\Sigma_a, \mathbf{R})$ is uniquely ergodic$\}$. Then S_4 is a residual subset of S.*

Proof. From [7, §4], we see that there is a residual subset $C_1 \subset C_0(\Omega)$ such that, if $b \in C_1$, then the equation $r(\omega \cdot t) - r(\omega) = \int_0^t b(\omega \cdot s)\, ds$ has no measurable solution. Hence, using 2.5, we see that S_4 is dense in S.

We now proceed as in [6]. Let $\{f_j\}_{j=1}^{\infty}$ be a countable dense subset of $C(\Sigma)$ (= the usual space of continuous, real-valued functions on Σ). Given $\varepsilon > 0$, let $V_{j,\varepsilon} = \{a \in S |$ there exists a constant $\alpha = \alpha(j)$ and a "time" $t \neq 0$ such that $|(1/t)\int_0^t f_j(\sigma \cdot s)\, ds - \alpha(j)| < \varepsilon$ for all $\sigma \in \Sigma\}$; here $\sigma \cdot s$ is the "position" of σ after time s under the flow $(\Sigma_a, \mathbf{R})$. Clearly $V_{j,\varepsilon}$ is open in S. It is not hard to see that $\bigcap_{j=1}^{\infty} \bigcap_{n=1}^{\infty} V_{j,1/n} = S_4$. We conclude that S_4 is a residual subset of S.

Finally, taking the intersection of S_3 and S_4, we obtain

3.10 Theorem. *There is a residual subset S_5 of S such that, if $a \in S_5$, then $(\Sigma_a, \mathbf{R})$ is minimal, strictly ergodic, and is a proximal extension of $(\Omega, \mathbf{R})$.*

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