A FIXED POINT FREE NONEXPANSIVE MAP

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ABSTRACT. In this note we give an example of a weakly compact convex subset of $L_1[0, 1]$ that fails to have the fixed point property for nonexpansive maps. This answers a long-standing question which was recently raised again by S. Reich [7].

1. Introduction. A (usually nonlinear) map $T$ on a subset $K$ of a Banach space $X$ is said to be nonexpansive if for every $k_1, k_2$ in $K$, $\|Tk_1 - Tk_2\| \leq \|k_1 - k_2\|$. Many authors have given conditions on the set $K$ that guarantee that a nonexpansive map $T$ on $K$ has a fixed point, e.g., [1], [2], [5], [6]. Usually $K$ is assumed to be weakly compact and convex. Of course, if $T$ is weakly continuous, then $T$ has a fixed point by the Schauder-Tychonoff fixed point theorem. For $T$ nonexpansive, (and not weakly continuous) positive results have been obtained only by placing additional requirements on $K$; however, it was unknown whether any of these additional requirements on $K$ were necessary. Our example shows that in fact some additional assumptions on $K$ are necessary.

2. The example. Let $X = L_1[0, 1]$ and let

$$K = \left\{ f \in L_1[0, 1]: \int_0^1 f = 1, 0 < f < 2, \text{a.e.} \right\}.$$ 

It is easy to see that $K$ is a weakly closed, convex subset of the order interval $\{ f: 0 < f < 2 \}$, and thus $K$ is weakly compact, because order intervals in $L_1[0, 1]$ are weakly compact. (This is a direct consequence of uniform integrability, [3, p. 292].) Define the map $T$ from $K$ to $K$ by

$$Tf(t) = \begin{cases} 2f(2t) \wedge 2, & 0 < t < \frac{1}{2}, \\ [2f(2t - 1) - 2] \vee 0, & \frac{1}{2} < t < 1. \end{cases}$$

(We will use equality throughout with the understanding that there may be an exceptional set of measure zero.) We leave it to the reader to check that $T$ is an isometry on $K$. 

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Suppose that $T$ has a fixed point $g$. We note first that $g = 21A$ for some set $A$ of measure one-half. Indeed,

$$\{ t: g(t) = 2 \} = \{ t: Tg(t) = 2 \}$$

$$= \{ t/2: g(t) = 2 \} + \left\{ \frac{1 + t}{2}: g(t) = 2 \right\}$$

$$+ \{ t/2: 1 < g(t) < 2 \}.$$  

(We are using $\oplus$ to denote disjoint union.) Because the measure of $\{ t/2: g(t) = 2 \} + \{ (1 + t)/2: g(t) = 2 \}$ is equal to the measure of $\{ t: g(t) = 2 \}$, it follows that $\{ t: 1 < g(t) < 2 \}$ is of measure zero. Iteration of this argument shows that

$$\{ t: 0 < g(t) < 2 \} = \bigcup_{n=0}^{\infty} \{ t: 2^{-n} < g(t) < 2^{-n+1} \}$$

is of measure zero, as well.

Next observe that for $g = 21A$

$$\{ t: T^n g(t) = 2 \} = \sum_{\epsilon_i \in \{0, 1\}} \left\{ \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots + \frac{\epsilon_n}{2^n} + \frac{t}{2^n}: t \in A \right\}$$

for all $n$. We have this for $n = 1$ above, and induction establishes it in general. Because $g$ is fixed, $A = \{ t: T^n g(t) = 2 \}$ for all natural numbers $n$ and thus, the intersection of $A$ with any interval with dyadic end points has measure exactly half the measure of the interval. Obviously no such measurable set exists. This contradiction shows that $T$ has no fixed point.

**Remark 1.** The set $K$ has diameter two, but $\|f - 1\| < 1$ for all $f \in K$ and thus, $K$ cannot be the minimal weakly compact convex subset invariant under $T$. In particular, the set

$$\bigcap_{i=1}^{\infty} \{ f: \|f - (1 + r_i)\| < 1 \} \cap \{ f: \|f - 1\| < 1 \} \cap K,$$

where $r_i = \text{sgn} \sin 2\pi i t$, the $i$th Rademacher function, is invariant.

**Remark 2.** It remains open whether there is a closed, bounded, convex subset of a reflexive space (hence, weakly compact) without the fixed point property for nonexpansive maps.

**Remark 3.** When viewed as a transformation acting on the sets $\{(x, y): 0 < y < f(x)\}$. This example is essentially the baker’s transformation from ergodic theory [4]. The various properties of our example can be derived from the well-known properties of that transformation.

**References**


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