HOMOGENEOUS TREE-LIKE CONTINUA

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Abstract. We prove that every k-junctioned homogeneous tree-like continuum is chainable, and hence a pseudo-arc. Possible extensions of this result are briefly discussed.

In 1959, Bing [B] proved that the pseudo-arc is the only homogeneous, chainable continuum. No other homogeneous tree-like continuum is known. We prove that if M is a k-junctioned homogeneous tree-like continuum, then M is chainable (and hence a pseudo-arc).

Burgess [Bu] has shown that every proper subcontinuum of a homogeneous k-junctioned tree-like continuum is a pseudo-arc. Extensive use will be made of this fact. We will also use the following theorem, proven by Hagopian [H], which follows from a result of Effros [E].

Theorem. Let M be a homogeneous continuum, and ε > 0. There exists δ > 0 such that if x, y ∈ M and dist(x, y) < δ, there is a homeomorphism h of M with h(x) = y and dist(z, h(z)) < ε for each z ∈ M. □

If P is a continuum and C is a chain, we will say that C essentially covers P provided C covers P but no proper subchain of C covers P. Other terminology (chain, pattern, amalgamation, etc.) and facts about hereditarily indecomposable continua which we use are standard. As usual, if H is a collection of sets then H* is the union of the elements of H.

We will now proceed directly to the proof of the main theorem.

Theorem. Every k-junctioned homogeneous tree-like continuum M is chainable.

Proof. Let U be a tree covering of mesh less than ε covering M. Let δ be a Lebesgue number for U which is also smaller than the distance between nonintersecting links of U. Choose γ > 0 such that, if x, y ∈ M and dist(x, y) < γ, there exists a homeomorphism h of M with h(x) = y and dist(z, h(z)) < δ/15k for each z ∈ M.

Let V be a k-junctioned tree chain of mesh less than γ which refines U and covers M. Let A be the collection of chains in V which are maximal with respect to...
containing no junction link of $V$ as an interior link. For each $\alpha \in A$, let $P_\alpha$ be a pseudo-arc in $M$ which is essentially covered by $\alpha$ and intersects no links of $V$ not in $\alpha$.

Let $P^0_\alpha$ and $P^1_\alpha$ be points of $P_\alpha$, in different composants, in the opposite end links of $\alpha$. If $\alpha \cap \alpha' \neq \emptyset$, and $P^0_\alpha, P^1_\alpha$ are in the common link of $\alpha$ and $\alpha'$, then there is a homeomorphism $h_{(\alpha, \alpha')}$ of $M$, moving no point more than $\delta/15k$, with $h_{(\alpha, \alpha')}(P^0_\alpha) = P^1_\alpha$. By the hereditary indecomposability of $M$, $h_{(\alpha, \alpha')}(P_\alpha) \subset P_\alpha$ or $h_{(\alpha, \alpha')}(P_\alpha) \supset P_\alpha$.

By composing the $h_{(\alpha, \alpha')}$'s (or their inverses) we can obtain $\tilde{\alpha} \in A$ and homeomorphisms $h_\alpha$, each moving no point more than $2\delta/15$, such that $h_\alpha(P_\alpha) \subset P_\alpha$ for each $\alpha \in A$.

We shall use these homeomorphisms and the pattern followed by $\tilde{\alpha}$ in $U$ to construct an $\varepsilon$-chain covering $M$ (and refining $U$).

Let $A_0 = \{ \tilde{\alpha} \}$ and, for each $i \in \omega_0$, let $A_{i+1} = \{ \alpha \in A | \alpha \cap (A^*_\alpha) \neq \emptyset, \text{ but } \alpha \notin A_i \}$. Let $g$ be a pattern which $\tilde{\alpha}$ follows in $U$, chosen such that if $g(\alpha) = \beta$ then the $\delta/3$-neighborhood of $\alpha$ is contained in $\beta$.

We will modify the chains $\alpha$ slightly before doing any amalgamation. If $\alpha \in A_i$ and $a$ is an end link of $\alpha$ such that every other chain $\alpha' \in A$ containing $a$ satisfies $\alpha' \in A_{i+1}$, then split $a$ into links, one, $L_\alpha$, for $\alpha$ and one, $L'_\alpha$, for each other $\alpha'$ containing $a$, such that $P_\alpha \cap L_\alpha = \emptyset$ for each $\alpha'$, $L_\alpha \cap P_\alpha = \emptyset$ for each $\alpha'$, and $L_\alpha \cap L'_\alpha = \emptyset$ for each distinct $\alpha', \alpha'' \in A_{i+1}$ containing $a$.

We will now amalgamate modifications of these altered chains $\alpha$ into a single chain $W$, of mesh less than $\varepsilon$, which follows the pattern $g$ in $U$. For each $\alpha$, let $C_\alpha$ be a chain covering $P_\alpha$ and refining $\alpha$ such that the image of each link of $C_\alpha$ under the homeomorphism $h_\alpha$ is a subset of a link of $\tilde{\alpha}$. We can choose $C_\alpha$ such that its boundary is contained only in its end links, which are in the end links of $\alpha$ and contain $P^0_\alpha$ and $P^1_\alpha$ respectively. Let $g_\alpha$ be a pattern, respecting end links, which $C_\alpha$ follows in $\alpha$. By using the fact that every proper subcontinuum of $M$ is a pseudo-arc, we can use the same type argument as used in the proof of Theorem 3 of [L] to show that the part of $M$ in the modified $\alpha$ can be amalgamated into a chain $D_\alpha$ such that (1) $D_\alpha$ follows the pattern $g_\alpha$ in $\alpha$, (2) for each $n$, the $n$th link of $C_\alpha$ is in the $n$th link of $D_\alpha$, and (3) the part of $M$ in the intersection of the link of the modified $\alpha$ containing $P_i^0 (i = 0, 1)$ with links not in the modified $\alpha$ is amalgamated into the same link of $D_\alpha$ as the point $P_i^0$.

The desired chain $W$ can now be constructed. For each $\alpha \in A \setminus \{ \tilde{\alpha} \}$ and positive integer $n$, choose a link $L_{\alpha,n}$ of $\tilde{\alpha}$ which contains the image under the homeomorphism $h_\alpha$ of the $n$th link of $C_\alpha$. If $b$ is a link of $\tilde{\alpha}$, the corresponding link of $W$ is $b$ together with the link of $D_\alpha$ containing the $n$th link of $C_\alpha$ where $b = L_{\alpha,n}$, for each $\alpha$ and each $n$.

Our choices of $g$ and of the $h_\alpha$'s guarantee that each link of $W$ is within $\delta/3$ of the link of $\tilde{\alpha}$ it contains. The correspondence between $D_\alpha$ and $C_\alpha$ guarantees that each amalgamation of $D_\alpha$ is a chain, and our choice of $h_\alpha$'s (obtained by composing $h_{(\alpha', \alpha)}$'s), modification of the end links of the $\alpha$'s, and construction of the $D_\alpha$'s guarantee that $D_\alpha$ and $D'_{\alpha'}$ together form a chain when amalgamated into $W$. □

The fact that $M$ was $k$-junctioned allowed us to form the $h_\alpha$'s such that none of them moved any point more than $2\delta/15$. Being $k$-junctioned also implied that $M$
was hereditarily indecomposable with every subcontinuum a pseudo-arc—which was crucial to our argument.

It is conceivable that a hereditarily indecomposable, non-\(k\)-junctioned, homogeneous tree-like continuum exists (perhaps a variation on Ingram's \([I]\) examples). Without chainable subcontinua, our techniques give one little to work with, even if one knows the continuum is hereditarily indecomposable.

Actually in the non-\(k\)-junctioned case, one does not know beforehand whether a homogeneous tree-like continuum must be hereditarily indecomposable. In fact it is still unknown whether a homogeneous tree-like continuum can contain an arc. It is known by a result of Jones \([J]\) that a homogeneous tree-like continuum must be indecomposable. Hagopian \([H2]\) and Jones \([J2]\) have shown that every homogeneous tree-like plane continuum is hereditarily indecomposable.

REFERENCES


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