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DIFFERENTIABILITY VIA ONE SIDED DIRECTIONAL DERIVATIVES

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ABSTRACT. Let F be a continuous mapping from an open subset D of a separable Banach space X into a Banach space Y. We show that if the one sided directional derivative $D_x^+F(a)$ of F at a in the direction x exists for each (a, x) from a dense G_δ subset S of an open set $D \times U \subset X \times X$, then F is Gâteaux differentiable on a dense G_δ subset of D. Similar results are obtained for Fréchet differentiability when X is finite-dimensional and for w^*-Gâteaux differentiability.

1. Introduction. Let X, Y be locally convex spaces, D be an open set in X and F: $D \to Y$ be a mapping. Let $a \in D$ and $x \in X$ be given. Then the following limit (if it exists)

$$\lim_{t \downarrow 0} \frac{1}{t}\left[F(a + tx) - F(a)\right]$$

is called the one sided directional derivative of F at a in the direction x and is denoted by $D_x^+F(a)$. If the mapping φ: $x \mapsto D_x^+F(a)$ is everywhere in X defined, linear, and continuous, then we say that F is Gâteaux differentiable at a. If moreover X is a normed space and the above limit exists uniformly with respect to x ranging over the unit ball of X, then F is said to be Fréchet differentiable at a.

In their joint work [3], K. S. Lau and C. E. Weil derived results concerning the set of points of differentiability of a continuous mapping from an information about two sided directional derivatives. The aim of this note is to show that the same conclusions can be obtained from weakened assumptions. Namely, it suffices to consider one sided directional derivatives only. The proofs are, except the proof of Theorem 3.4, slight refinements of those in the quoted paper [3].

2. Topological tools. The concept of Baire space will play the crucial role throughout the whole note. A Hausdorff topological space X is called a Baire space if the intersection of any sequence of open dense subsets in X is dense in X. Let us recall that open as well as dense G_δ subsets of a complete metric space are Baire spaces.

From [3] we shall borrow the following three lemmas, Lemma 2.2 being a slight generalization of [3, Lemma 2.3].

LEMMA 2.1. *Let X and Y be Baire spaces with Y second countable and let E be a dense G_δ subset of $X \times Y$.*

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Then there exists a dense G_δ subset B of X such that the set

$$E_a = \{y \in Y \mid (a, y) \in E\}$$

is dense G_δ in Y whenever $a \in B$.

LEMMA 2.2. *Let X be a Baire space, $r\colon X \to (0, +\infty)$ be a lower semicontinuous function and (Y, ρ) be a metric space. Denote $\Omega = \{(x, t) \in X \times \mathbf{R} \mid t \in (0, r(x))\}$ and let $F\colon \Omega \to Y$ be a given mapping. Assume that, for each $(x, t) \in \Omega$, the mapping $u \mapsto F(u, t)$ is continuous at $u = x$ and that there exists a dense G_δ subset A in X such that the limit $F_0(a) = \lim_{t \downarrow 0} F(a, t)$ exists for all $a \in A$.*

Then there exists a dense G_δ subset E in X such that

$$\forall a \in E \quad F(x, t) \to F_0(a) \quad \text{as } (x, t) \to (a, 0), \qquad t > 0.$$

PROOF. For $m, n = 1, 2, \ldots,$ set

$$A_{mn} = \{x \in X \mid \rho(F(x, t), F(x, s)) \leqslant 1/m \text{ whenever } t, s \in (0, \min(1/n, r(x)))\}.$$

As the mapping $u \mapsto F(u, t)$ is continuous at x and r is lower semicontinuous, the sets A_{mn} are closed. Moreover, from the hypotheses we know that $A \subset \bigcup_{n=1}^{\infty} A_{mn}$. Hence the set $G_m = \bigcup_{n=1}^{\infty} A_{mn}^0$ is open dense in X. Now put $E = A \cap \bigcap_{m=1}^{\infty} G_m$. This set is dense G_δ in X and we shall show that it has the required property.

So fix $a \in E$ and let $\varepsilon > 0$ be given. Take $m > 3/\varepsilon$. As $a \in G_m$, there is n such that $a \in A_{mn}^0$. Since r is lower semicontinuous, there is a neighbourhood V of a such that $\frac{1}{2} r(a) < r(x)$ for $x \in V$. Thus, for all $x \in V \cap A_{mn}^0$ and all $s, t \in (0, \min(1/n, \frac{1}{2} r(a)))$, we have $\rho(F(x, s), F(x, t)) \leqslant 1/m$, especially $\rho(F(a, s), F_0(a)) \leqslant 1/m$. Fix one such s and, from the continuity of the mapping $u \mapsto F(u, s)$ at $u = a$, find a neighbourhood U of a such that $\rho(F(x, s), F(a, s)) < 1/m$ whenever $x \in U$. Finally, putting $W = U \cap V \cap A_{mn}^0$, we get that, for all $x \in W$ and all $t \in (0, \min(1/n, \frac{1}{2} r(a)))$,

$$\begin{aligned} \rho(F(x, t), F_0(a)) &\leqslant \rho(F(x, t), F(x, s)) + \rho(F(x, s), F(a, s)) \\ &\quad + \rho(F(a, s), F_0(a)) < 1/m + 1/m + 1/m < \varepsilon. \end{aligned}$$

LEMMA 2.3. *Let X be a Banach space and let A be a dense G_δ subset of an open set $U \subset X$.*

Then the linear span of A is the whole space X.

3. Theorems.

THEOREM 3.1. *Let X, Y be real Banach spaces, with X separable, D, U be nonempty open subsets of X and $F\colon D \to Y$ be a continuous mapping. Suppose that there exists a dense G_δ subset S of $D \times U$ such that*

$$\forall (a, x) \in S \quad D_x^+ F(a) = \lim_{t \downarrow 0} \frac{1}{t} \left[F(a + tx) - F(a) \right] \text{ exists.}$$

Then F is Gâteaux differentiable on a dense G_δ subset of D.

PROOF. Define $r\colon D \times U \to (0, +\infty)$ by

$$r(a, x) = \sup\{t \in (0, 1] \mid a + (0, t)x \subset D\}, \qquad (a, x) \in D \times U.$$

Clearly, r is a lower semicontinuous function. Put also

$$\Omega = \{(a, x, t) \in D \times U \times \mathbf{R} \mid t \in (0, r(a, x))\}$$

and define the mapping $\tilde{F}\colon \Omega \to Y$ by

$$\tilde{F}(a, x, t) = \frac{1}{t}[F(a + tx) - F(a)], \qquad (a, x, t) \in \Omega.$$

By the hypothesis,

$$\forall (a, x) \in S \quad \tilde{F}(a, x, t) \to D_x^+F(a) \quad \text{as } t \downarrow 0.$$

Hence, according to Lemma 2.2, there is a dense G_δ subset E of $D \times U$ such that

$$\forall (a, x) \in E \quad \tilde{F}(a', x', t) \to D_x^+F(a) \quad \text{as } (a', x', t) \to (a, x, 0), \qquad t > 0. \tag{3.1}$$

Further, by Lemma 2.1, there is a dense G_δ subset B of D such that the set $E_a = \{x \in U \mid (a, x) \in E\}$ is dense G_δ in U whenever $a \in B$.

Next we shall show that F is Gâteaux differentiable at each $a \in B$, which will complete the proof. So fix $a \in B$. Let M denote the set of all $x \in X$ such that

$$\tilde{F}(a', x, t) \to D_x^+F(a) \quad \text{as } (a', t) \to (a, 0), \qquad t > 0. \tag{3.2}$$

Let $x, y \in X$ be given. For a' near a and small $t > 0$, we have

$$\begin{aligned}\tilde{F}(a', x + y, t) &= \frac{1}{t}[F(a' + t(x + y)) - F(a')] \\ &= \frac{1}{t}[F((a' + ty) + tx) - F(a' + ty)] + \frac{1}{t}[F(a' + ty) - F(a')] \\ &= \tilde{F}(a' + ty, x, t) + \tilde{F}(a', y, t).\end{aligned}$$

Hence, if $x, y \in M$, then (3.2) yields that $x + y \in M$ and

$$D_{x+y}^+F(a) = D_x^+F(a) + D_y^+F(a).$$

Also, as

$$\tilde{F}(a', -x, t) = -\tilde{F}(a' - tx, x, t),$$

it follows that $x \in M$ implies $-x \in M$ and

$$D_{-x}^+F(a) = -D_x^+F(a).$$

Moreover, for $\tau > 0$, we have

$$\tilde{F}(a', \tau x, t) = \tau\tilde{F}(a', x, \tau t).$$

Hence, if $x \in M$ and $\tau > 0$, then $\tau x \in M$ and

$$D_{\tau x}^+F(a) = \tau D_x^+F(a).$$

We have thus verified that M is a linear space over which the mapping φ: $x \mapsto D_x^+F(a)$ is linear. But, by (3.1), M contains E_a, which is dense G_δ in D. Therefore Lemma 2.3 yields that $M = X$.

It remains to prove the continuity of the mapping φ. Let $r > 0$ be so small that $a + rV \subset D$, where V is the closed unit ball in X. Denote, for $k = 1, 2, \ldots,$

$$A_k = \{x \in V \mid \|F(a + tx) - F(a)\| < kt \text{ whenever } t \in (0, \min(1/k, r))\}.$$

As F is continuous, A_k are closed. And since $\bigcup_{k=1}^{\infty} A_k = V$, there is some k such that $A_k^0 \neq \varnothing$. Thus, because

$$\forall x \in A_k^0 \quad \|\varphi(x)\| = \|D_x^+F(a)\| = \lim_{t\downarrow 0} \frac{1}{t}\|F(a + tx) - F(a)\| < k$$

and φ is linear, φ is also continuous.

If f is a convex continuous function defined on an open subset D of a separable Banach space X, then $D_x^+f(a)$ exists for all $(a, x) \in D \times X$. Thus we get from the above theorem the result of S. Mazur [4] asserting that f is Gâteaux differentiable on a dense G_δ subset of D.

THEOREM 3.2. *Let $\{e_1, \ldots, e_n\}$ be the canonical basis in $\mathbf{R}^n$ and let $F: D \to Y$ be a continuous mapping from an open subset D of $\mathbf{R}^n$ into a Banach space Y. Assume that there is a dense G_δ subset A in D such that, for each $a \in A$ and each $i = 1, \ldots, n$, $D_{e_i}^+F(a)$ exists.*

Then F is Fréchet differentiable on a dense G_δ subset E in D.

PROOF. For brevity write $D_i^+F(a) = D_{e_i}^+F(a)$. Denoting

$$r(a) = \sup\{t \in (0, 1] \mid a + (-t, t)e_i \subset D, i = 1, \ldots, n\}, \qquad a \in D,$$

r is a lower semicontinuous function. Put also

$$\Omega = \{(a, t) \in D \times \mathbf{R} \mid |t| \in (0, r(a))\}$$

and define the mapping $\tilde{F}: \Omega \to Y^n$ by

$$\tilde{F}(a, t) = (\tilde{F}_1(a, t), \ldots, \tilde{F}_n(a, t)),$$

where

$$\tilde{F}_i(a, t) = \frac{1}{t}[F(a + te_i) - F(a)], \qquad (a, t) \in \Omega, i = 1, \ldots, n.$$

According to the assumption, we have that

$$\forall a \in A \quad \tilde{F}(a, t) \to (D_1^+F(a), \ldots, D_n^+F(a)) \quad \text{as } t\downarrow 0.$$

By Lemma 2.2, there is a dense G_δ subset E in D such that

$$\forall a \in E \quad \tilde{F}(a', t) \to (D_1^+F(a), \ldots, D_n^+F(a)) \quad \text{as } (a', t) \to (a, 0), \qquad t > 0. \tag{3.3}$$

We shall show that E is the set we are looking for. Noting that, for a' near to a and small $t > 0$,

$$\begin{aligned}\tilde{F}_i(a', -t) &= \frac{1}{-t}[F(a' - te_i) - F(a')] \\ &= \frac{1}{t}[F((a' - te_i) + te_i) - F(a' - te_i)] = \tilde{F}_i(a' - te_i, t),\end{aligned}$$

we get from (3.3) that

$$\forall a \in E \quad \tilde{F}_i(a', t) \to D_i^+F(a) \quad \text{as } (a', t) \to (a, 0), \qquad t \neq 0,$$

$i = 1, \ldots, n$. The rest of the proof is the same as in the proof of [3, Theorem 3.2] and hence is omitted.

It should be noted that, in Lemma 2.2 and Theorem 3.2, we may replace the text "a dense G_δ subset" by "a subset of second category" without changing their proofs. We obtain thus the result due to S. V. Gorlenko [2, Theorem 3]: If a continuous function on an open subset D of $\mathbf{R}^n$ has partial derivatives on a subset of second category, then it is Fréchet differentiable on a subset of second category.

It is easy to see that, under the assumptions of the above theorem, the mapping F is locally Lipschitzian at each point from an open dense subset of D, see [3]. Combining this fact with the Rademacher theorem [1, p. 218], we obtain

THEOREM 3.3. *Let F be a continuous mapping from an open set $D \subset \mathbf{R}^n$ into $\mathbf{R}^m$. Let there be a dense G_δ subset A of D such that all $D_{e_i}^+F(a)$, $i = 1, \ldots, n$, exist for each $a \in A$.*

Then F is Fréchet differentiable on a dense subset in D with positive (Lebesgue) measure.

Next let X, Y be Banach spaces, $D \subset X$ be open and $F\colon D \to (Y^*, w^*)$ be a mapping. Then $D_x^+F(a)$ means the weak* limit

$$w^* - \lim_{t \downarrow 0} \frac{1}{t}\left[F(a + tx) - F(a)\right].$$

Following [3], F is said to be w^*-Gâteaux differentiable at $a \in D$ if the mapping $x \mapsto D_x^+F(a)$ is everywhere defined, linear, and bounded.

THEOREM 3.4. *Let X, Y be real separable Banach spaces, D be an open subset of X and $F\colon D \to (Y^*, w^*)$ be a continuous mapping. Suppose there exists a dense G_δ subset A of D such that, for each $(a, x) \in A \times X$, $D_x^+F(a)$ exists.*

Then F is w^*-Gâteaux differentiable on a dense G_δ subset of D.

PROOF. Let $\{y_n\}$ be a dense sequence in Y and denote $f_n(x) = (F(x), y_n)$, where $(\cdot, \cdot)$ means the duality pairing between Y^* and Y. By Theorem 3.1, f_n is Gâteaux differentiable on some dense G_δ subset E_n of D. Putting $E = \bigcap_{n=1}^\infty E_n$, the last set is dense G_δ in D. We shall show that f is w^*-Gâteaux differentiable at each $a \in E$, which will complete the proof.

So fix $a \in E$. We know that the mappings $x \mapsto D_x^+f_n(a) = (D_x^+F(a), y_n)$ are linear on X. And as $\{y_n\}$ is dense in Y, the mapping $\varphi\colon x \mapsto D_x^+F(a)$ (which is everywhere defined by the assumption) is also linear. The boundedness of φ can be shown in the same way as in the proof of Theorem 3.1. Thus the mapping φ is everywhere defined, linear, and bounded for each $a \in E$, and the proof is completed.

It should be noted that, from Theorems 3.1–3.4, we can at once obtain [3, Theorems 3.1–3.4] where the two sided directional derivatives are considered.

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