SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

ON THE WEIGHT AND PSEUDOWEIGHT OF LINEARLY ORDERED TOPOLOGICAL SPACES

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Abstract. We derive a simple formula for the weight of a LOTS using the pseudoweight. As an application we give a very short proof of the nonorderability of the Sorgenfrey-line.

1. Definitions. Let \((X, \tau)\) be a \(T_1\)-space.
A collection \(\mathcal{U} \in \tau\) is called a \(\psi\)-base for \(X \ [3]\) if
(i) \(\mathcal{U}\) covers \(X\), and
(ii) \(\bigcap \{ U | x \in U \in \mathcal{U} \} = \{ x \}\), for all \(x \in X\).

We put as usual \(\psiw(X) = \min \{|\mathcal{U}| |\mathcal{U} \text{ is a } \psi\text{-base for } X\}\).

Recall that
\(c(X) = \sup \{|\mathcal{U}| |\mathcal{U} \text{ c } \tau \text{ and } \mathcal{U} \text{ is disjoint}\}\),
and
\(w(X) = \min \{|\mathcal{B}| |\mathcal{B} \subset \tau \text{ and } \mathcal{B} \text{ is a base for } X\}\).

2.

Theorem. If \(X\) is a Linearly Ordered Topological Space (LOTS), then \(w(X) = c(X) \cdot \psiw(X)\).

Proof. "\(>\)" is obvious.
"\(<\)". Let \(\mathcal{U} = \{ U_i \}_{i \in I} \) be a \(\psi\)-base for \(X\) with \(|\mathcal{U}| = \psiw(X)\). For each \(i \in I\) let \(\{ C_{ij} \}_{j \in J_i} \) be the collection of convex components of \(U_i\). Put \(\mathcal{B} = \{ C_{ij} | j \in J_i, i \in I \}\). Since \(|J_i| < c(X)\) for all \(i\), we see that \(|\mathcal{B}| < c(X) \cdot \psiw(X)\). We claim that \(\mathcal{B}\) is a subbase for \(X\).

Indeed, take \(x \in X\) and \((a, b) \ni x\). Since \(\bigcap \{ B | x \in B \in \mathcal{B} \} = \{ x \}\), there exist \(B_a, B_b \in \mathcal{B}\) such that \(x \in B_a \ni a\) and \(x \in B_b \ni b\). Then \(x \in B_a \cap B_b \ni (a, b)\).

\[\square\]

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3. Examples. We shall show that our theorem cannot be improved.

3.1. The Sorgenfrey-line $S$ shows that we cannot replace "$X$ is a LOTS" by "$X$ is a GO-space". Indeed $w(S) = 2^\omega$, $c(S) = \omega$ and $\psi w(S) = \omega$ [take all intervals with rational endpoints].

This shows once again that $S$ is not a LOTS. For other, more involved, proofs, see for example [1] and [4].

3.2. The largest "natural" invariant below $c(X)$ (in the case of LOTS) is $l(X)$, the Lindelöf number of $X$. We shall see that we cannot replace $c$ by $l$:

Let $A \subset \mathbb{R}$ be a subset of cardinality $2^n$ with the property that for any closed set $C \subset \mathbb{R}$ with either $C \subset A$ or $C \subset \mathbb{R} \setminus A$ we have that $C$ is countable.

Build a Michael-line $M(A)$ by isolating every $a \in A$. The resulting space is Lindelöf [6]. Now let $X$ be the associated LOTS of the GO-space $M(A)$ [5].

$X = \{(x, n) \in \mathbb{R} \times \mathbb{Z} | x \in \mathbb{R} \setminus A \Rightarrow n = 0\}$ endowed with the lexicographic order. The map $f: X \to M(A)$ defined by $f((x, n)) = x$ is a retraction with countable fibers; hence $X$ is Lindelöf.

Let us put $U_q = \{x \in X | x < (q, 0)\}$, $V_q = \{x \in X | x > (q, 0)\}$ and $O_n = \{(a, n) | a \in A\}$. Then $\mathcal{U} = \{U_q\}_{q \in \mathbb{Q}} \cup \{V_q\}_{q \in \mathbb{Q}} \cup \{O_n\}_{n \in \mathbb{Z}}$ is a countable $\psi$-base for $X$. Finally we have $w(X) = |A| = 2^\omega$.

3.3. The largest invariant—to our knowledge—below $\psi w(X)$ is $psw(X) = \min\{\text{ord}(\mathcal{U}) | \mathcal{U} \text{ is a } \psi\text{-base for } X\}$ [2]. If we let $Z$ be a dense left-separated subspace of a connected Souslin-line, then $Z$ is a LOTS with a point-countable base [7]. Thus we have $w(Z) = \omega_1 > \omega = c(Z) \cdot psw(Z)$; hence $\psi w$ cannot be replaced by $psw$.

REFERENCES