CONCAVITY OF POWERS OF A CONVOLUTION

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Abstract. A short geometric proof of a result of Brascamp and Lieb on concavity of powers of a convolution is given.

The following theorem, a special case of Corollary 3.5 of [2], has a very simple proof for positive integers \( p \) and \( q \).

Let \( f, g : \mathbb{R}^n \to \mathbb{R}^+ \cup \{0\} \) be positive on bounded convex sets \( S(f) \), \( S(g) \) respectively and 0 elsewhere. Let \( p \) and \( q \) be positive integers.

**Theorem 1.** If \( f^{1/p} \) and \( g^{1/q} \) are concave on their respective, convex supports then \( (f \ast g)^{1/(p+q+n)} \) is concave on the Minkowski set sum \( S(f) + S(g) \) and 0 elsewhere.

**Remark.** Suitably interpreted this also holds if \( p \) or \( q \), or both, are zero. The exponent \( 1/(p + q + n) \) is best possible, in all cases.

**Proof of Theorem 1.** Let \( F \subseteq \mathbb{R}^{p+n} = \{(x_1, x_2, \ldots, x_n, w_1, w_2, \ldots, w_p): \left| w_i \right| < \frac{1}{2} f^{1/p}(x_1, x_2, \ldots, x_n) \text{ for } 1 \leq i \leq p\} \). Let

\[
G \subseteq \mathbb{R}^{q+n} = \{(y_1, y_2, \ldots, y_n, v_1, v_2, \ldots, v_q): \left| v_i \right| < \frac{1}{2} g^{1/q}(y_1, y_2, \ldots, y_n) \text{ for } 1 \leq i \leq q\}.
\]

Then \( F, G \) and \( F \times G \) are convex sets.

Denote \((x_1, \ldots, x_n, w_1, \ldots, w_p, y_1, \ldots, y_n, v_1, \ldots, v_q)\) by \((\xi, \omega, \eta, \nu)\) and for \( \xi \in \mathbb{R}^n \) let \( H_\xi \) be the affine space \( \{(\xi, \omega, \eta, \nu): \xi + \eta = \xi\} \). Let \( J_\eta \) be the affine space \( \{(\xi, \omega, \eta, \nu): \xi = \eta\} \). For arbitrary \( \xi \) and \( \eta \), \( F \times G \cap H_\xi \cap J_\eta \) is either empty or the Cartesian product of a \( p \)-dimensional open cube of volume \( f(\eta) \) with a \( q \)-dimensional open cube of volume \( g(\xi - \eta) \). In either case the \((p + q)\)-dimensional volume of \( H_\xi \cap J_\eta \cap F \times G \) is \( f(\eta)g(\xi - \eta) \).

Since \( H_\xi \) is the disjoint union of the \( H_\xi \) and the distance between flats \( H_\xi \cap J_\xi \) and \( H_\xi \cap J_\eta \) is \( \sqrt{2} ||\xi - \eta|| \), the \((p + q + n)\)-dimensional volume \( V(\xi) \) of \( F \times G \cap H_\xi \) is given by

\[
V(\xi) = (\sqrt{2})^n \int_{\xi \in \mathbb{R}^n} f(\eta)g(\xi - \eta) \, d\eta = (\sqrt{2})^n (f \ast g)(\xi).
\]

Let us fix \( z_2, \ldots, z_n \) and consider \( V(\xi) \) as a function of \( z_1 \) alone.

Since the choice of coordinate axes is arbitrary it will suffice to prove that \( V(\xi)^{1/(p+q+n)} \) is concave in \( z_1 \) on the interval where it is positive.

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Let $H' = \{ (\bar{x}, \bar{w}, \bar{y}, \bar{v}) : x_i + y_i = z_i \text{ for } 2 \leq i \leq n \}$, and let $H''(z_1) = \{ (\bar{x}, \bar{w}, \bar{y}, \bar{v}) : x_1 + y_1 = z_1 \}$. Then $F \times G \cap H'$ is convex and $H''(z_1)$ is a hyperplane in $H'$. Let $\phi(z_1) = V^{1/(p+q+n)}$ where $V$ is the $(p + q + n)$-dimensional volume of $(F \times G \cap H') \cap H''(z_1)$. Then by the Brunn-Minkowski theorem, $\phi(z_1)$ is concave on its support. □

REFERENCES


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