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CONCAVITY OF POWERS OF A CONVOLUTION

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ABSTRACT. A short geometric proof of a result of Brascamp and Lieb on concavity of powers of a convolution is given.

The following theorem, a special case of Corollary 3.5 of [2], has a very simple proof for positive integers p and q.

Let $f,\ g$: $\mathbf{R}^n \to \mathbf{R}^+ \cup \{0\}$ be positive on bounded convex sets $S(f)$, $S(g)$ respectively and 0 elsewhere. Let p and q be positive integers.

THEOREM 1. *If $f^{1/p}$ and $g^{1/q}$ are concave on their respective, convex supports then $(f * g)^{1/(p+q+n)}$ is concave on the Minkowski set sum $S(f) + S(g)$ and 0 elsewhere.*

REMARK. Suitably interpreted this also holds if p or q, or both, are zero. The exponent $1/(p + q + n)$ is best possible, in all cases.

PROOF OF THEOREM 1. Let $F \subseteq \mathbf{R}^{p+n} = \{(x_1, x_2 \cdots x_n, w_1, w_2 \cdots w_p)\colon |w_i| < \frac{1}{2}f^{1/p}(x_1, x_2 \cdots x_n) \text{ for } 1 \leqslant i \leqslant p\}$. Let

$$G \subseteq \mathbf{R}^{q+n} = \left\{(y_1, y_2 \cdots y_n, v_1, v_2 \cdots v_q)\colon |v_i| < \tfrac{1}{2}g^{1/q}(y_1 \cdots y_n) \text{ for } 1 \leqslant i \leqslant q\right\}.$$

Then F, G and $F \times G$ are convex sets.

Denote $(x_1 \cdots x_n, w_1 \cdots w_p, y_1 \cdots y_n, v_1 \cdots v_q)$ by $(\bar{x}, \bar{w}, \bar{y}, \bar{v})$ and for $\bar{z} \in \mathbf{R}^n$ let $H_{\bar{z}}$ be the affine space $\{(\bar{x}, \bar{w}, \bar{y}, \bar{v})\colon \bar{x} + \bar{y} = \bar{z}\}$. Let $J_{\bar{\alpha}}$ be the affine space $\{(\bar{x}, \bar{w}, \bar{y}, \bar{v})\colon \bar{x} = \bar{\alpha}\}$. For arbitrary $\bar{z}$ and $\bar{\alpha}$, $F \times G \cap H_{\bar{z}} \cap J_{\bar{\alpha}}$ is either empty or the Cartesian product of a p-dimensional open cube of volume $f(\bar{\alpha})$ with a q-dimensional open cube of volume $g(\bar{z} - \bar{\alpha})$. In either case the $(p + q)$-dimensional volume of $H_{\bar{z}} \cap J_{\bar{\alpha}} \cap F \times G$ is $f(\bar{\alpha})g(\bar{z} - \bar{\alpha})$.

Since $H_{\bar{z}}$ is the disjoint union of the $H_{\bar{z}} \cap J_{\bar{\alpha}}$ and the distance between flats $H_{\bar{z}} \cap J_{\bar{\alpha}}$ and $H_{\bar{z}} \cap J_{\bar{\alpha}'}$ is $\sqrt{2}\,\|\bar{\alpha} - \bar{\alpha}'\|$, the $(p + q + n)$-dimensional volume $V(\bar{z})$ of $F \times G \cap H_{\bar{z}}$ is given by

$$V(z) = (\sqrt{2}\,)^n \int_{\bar{\alpha} \in \mathbf{R}^n} f(\bar{\alpha})g(\bar{z} - \bar{\alpha})\, d\bar{\alpha} = (\sqrt{2}\,)^n (f * g)(\bar{z}).$$

Let us fix $z_2, \ldots, z_n$ and consider $V(\bar{z})$ as a function of z_1 alone.

Since the choice of coordinate axes is arbitrary it will suffice to prove that $V(\bar{z})^{1/(p+q+n)}$ is concave in z_1 on the interval where it is positive.

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Let $H' = \{(\bar{x}, \bar{w}, \bar{y}, \bar{v})\colon x_i + y_i = z_i \text{ for } 2 < i < n\}$, and let $H''(z_1) = \{(\bar{x}, \bar{w}, \bar{y}, \bar{v})\colon x_1 + y_1 = z_1\}$. Then $F \times G \cap H'$ is convex and $H''(z_1)$ is a hyperplane in H'. Let $\phi(z_1) = V^{1/(p+q+n)}$ where V is the $(p + q + n)$-dimensional volume of $(F \times G \cap H') \cap H''(z_1)$. Then by the Brunn-Minkowski theorem, $\phi(z_1)$ is concave on its support. □

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