A RESULT RELATED TO A THEOREM BY PIANIGIANI

ABRAHAM BOYARSKY\(^1\) AND GABRIEL HADDAD

Abstract. Let \(\tau: J \rightarrow J\) be a piecewise \(C^2\) map, where \(J\) is an interval, satisfying \(\inf|\tau'| > 1\). An upper bound for the number of independent absolutely continuous measures invariant under \(\tau\) is presented.

Introduction. Let \(J = [a, b]\) be an interval, \(\mathcal{B}\) the Lebesgue measurable subsets of \(J\), and \(\lambda\) the Lebesgue measure on \(J\). Let \(\tau: J \rightarrow J\) be a piecewise \(C^2\) transformation satisfying \(\inf|\tau'(x)| > 1\) where the derivative exists. In [1] it is shown that \(\tau\) admits an absolutely continuous invariant measure \(\mu\), i.e., \(\mu(A) = \mu(\tau^{-1}(A))\) for all \(A \in \mathcal{B}\), and

\[
\mu(A) = \int_A f\,d\lambda,
\]

where we refer to \(f\) as the density invariant under \(\tau\). Clearly \(f > 0\) and \(f \in L_1\), the space of integrable functions on \(J\).

Let \(\mathcal{F}_\tau\) denote the space of densities invariant under \(\tau\) and \(\{a_1, a_2, \ldots, a_k\}\) those points in \(J\) where \(\tau'\) does not exist. The main result of [2] asserts that \(\dim \mathcal{F}_\tau < k\). In fact it is very easy to establish a better bound. Let \(a = b_0 < b_1 < \cdots < b_m < b_{m+1} = b\) be the partition of \(J\) such that \(\tau\) is continuous and monotonic on each interval \((a_j, b_j)\). Clearly \(m < k\), and \(\dim \mathcal{F}_\tau < m\). In the special case where \(\tau\) is continuous on \(J\), the total number of peaks and valleys in the graph of \(\tau\) constitutes an upper bound for \(\dim \mathcal{F}_\tau\).

In §3 of [3] a still better bound is established for \(\dim \mathcal{F}_\tau\). Let \(\{b_1, b_2, \ldots, b_m\}\) be the partition defined in the previous paragraph. For each \(1 < j < m\), define the pair

\[
\langle u_j, v_j \rangle = \langle \tau(b_j^-), \tau(b_j^+) \rangle,
\]

where \(u_j\) is regarded as \(u_j^+\) or \(u_j^-\) depending on whether \(\tau(a_j - \varepsilon) > u_j\) or \(\tau(a_j - \varepsilon) < u_j\).

Two pairs \(\langle u_i, v_i \rangle\) and \(\langle u_j, v_j \rangle\) are said to be dependent if they have one or both coordinates in common. Otherwise the pairs are independent. Let \(N_\tau\) denote the maximal number of independent pairs. Then Theorem 2 of [3] asserts that \(\dim \mathcal{F}_\tau < N_\tau\). In this note we suggest a modified definition of dependence and present a different bound for the number of absolutely continuous measures invariant under \(\tau\).

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2. Dependence of densities. Let \( \tau: J \to J \) be piecewise \( C^2 \) satisfying \( \inf |\tau'(x)| > 1 \) and let \( \mathcal{P} = \{ b_1, b_2, \ldots, b_m \} \) be the partition on which \( \tau \) is piecewise continuous and monotonic. We shall say that \( b_i \) and \( b_j \) are dependent if
\[
\tau(b_i - e, b_i + e) \cap \tau(b_j - e, b_j + e)
\]
has positive measure for every \( e > 0 \). This implies, but is not equivalent to
\[
\langle \tau(b_i^-), \tau(b_j^+) \rangle \cap \langle \tau(b_i^+), \tau(b_j^-) \rangle \neq \emptyset.
\]
This definition of dependence for a pair of discontinuities in \( \mathcal{P} \) is reflexive, symmetric, but not transitive. A collection \( S \subset \mathcal{P} \) is said to be dependent if every pair of points in this collection is dependent, and maximal if \( S \) is not a proper subset of any dependent collection. Notice that two distinct maximal dependent collections may have nonempty intersection, and such a collection may consist of a single point. Thus, given \( b_j \in \mathcal{P} \), there exists at least one and at most two maximal dependent collections containing \( b_j \). In particular, when \( \tau \) is continuous at \( b_j \), there exists only one maximal dependent collection containing this point. Let \( H_\tau \) be the number of distinct maximal dependent collections. Then, we have

**Theorem.** \( \dim \mathcal{F}_\tau < H_\tau \).

**Proof.** We first show that if \( f_1 \) and \( f_2 \) are invariant with disjoint supports, then to each \( f_i \) there corresponds one maximal dependent collection \( S_i \) and \( S_1 \neq S_2 \). Letting \( M_i = \text{spt} f_i \), it is easy to see that \( \text{int} M_i \) has to contain at least one point of \( \mathcal{P} \), say \( b'_j \). Let \( S_1 \) and \( S_2 \) be any maximal collections containing \( b'_1 \) and \( b'_2 \), respectively, and suppose \( S_1 = S_2 \). Then \( b'_1 \) and \( b'_2 \) are dependent. Since \( \tau(M_i) \subset M_i \) a.e. \([1]\), and \( (b'_1 - e, b'_1 + e) \subset M_i \) for some \( e < 0 \), the dependence of \( b'_1 \) and \( b'_2 \) implies
\[
\lambda(M_1 \cap M_2) > \lambda[\tau(b'_1^-, b'_1 + e) \cap \tau(b'_2^- - e, b'_2 + e)] > 0.
\]
This is a contradiction. Therefore, \( S_1 \) and \( S_2 \) must be distinct.

Now let \( \{ f_1, f_2, \ldots, f_n \} \) be a maximal set of disjoint densities invariant under \( \tau \) \([2]\). By the preceding argument we see that there exists a 1-1 mapping from \( \{ f_1, \ldots, f_n \} \) into \( \{ S_1, \ldots, S_{H_\tau} \} \). Thus \( n < H_\tau \). Q.E.D.

3. Examples. (a) Consider the transformation \( \tau \) shown in Figure 1.

![Figure 1](image)

We see that \( \{ b_1, b_2, b_3 \} \) is the unique collection which is dependent and maximal. Thus \( H_\tau = 1 \) and there exists a unique absolutely continuous measure invariant under \( \tau \). The bound from \([2]\) is 8, since there are 8 discontinuities in \( \tau' \) in \((0, 1)\).
(b) Let $\tau$ have the graph shown in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Figure 2}
\end{figure}

For each discontinuity, we give the corresponding maximal dependent collection or collections as the case may be:

- $b_1$: $\{b_1, b_3, b_3\}$ and $\{b_1, b_4\}$,
- $b_2$: $\{b_2, b_4, b_3\}$,
- $b_3$: $\{b_1, b_3, b_3\}$,
- $b_4$: $\{b_1, b_4\}$ and $\{b_2, b_4, b_3\}$,
- $b_5$: $\{b_1, b_3, b_3\}$ and $\{b_2, b_4, b_3\}$,
- $b_6$: $\{b_6\}$.

There are 4 independent collections. Therefore $\tau$ admits at most four independent invariant densities.

Notice that for this example the bound of [2] is 7, since there are 7 discontinuities of $\tau'$ in $(0, 1)$.

\textbf{References}


\textbf{Department of Mathematics, Sir George Williams Campus, Concordia University, Montreal, Quebec, Canada H3G 1M8}