**Abstract.** Let \( r: J \to J \) be a piecewise \( C^2 \) map, where \( J \) is an interval, satisfying \( \inf|r'| > 1 \). An upper bound for the number of independent absolutely continuous measures invariant under \( r \) is presented.

**Introduction.** Let \( J = [a, b] \) be an interval, \( \mathcal{B} \) the Lebesgue measurable subsets of \( J \), and \( \lambda \) the Lebesgue measure on \( J \). Let \( r: J \to J \) be a piecewise \( C^2 \) transformation satisfying \( \inf|\tau'(x)| > 1 \) where the derivative exists. In [1] it is shown that \( \tau \) admits an absolutely continuous invariant measure \( \mu \), i.e., \( \mu(A) = \mu(\tau^{-1}(A)) \) for all \( A \in \mathcal{B} \), and

\[
\mu(A) = \int_A f \, d\lambda,
\]

where we refer to \( f \) as the density invariant under \( \tau \). Clearly \( f > 0 \) and \( f \in \mathcal{C}_1 \), the space of integrable functions on \( J \).

Let \( \mathcal{F}_r \) denote the space of densities invariant under \( \tau \) and \( \{a_1, a_2, \ldots, a_k\} \) those points in \( J \) where \( \tau' \) does not exist. The main result of [2] asserts that \( \dim \mathcal{F}_r < k \).

In fact it is very easy to establish a better bound. Let \( a = b_0 < b_1 < \cdots < b_m < b_{m+1} = b \) be the partition of \( J \) such that \( \tau \) is continuous and monotonic on each interval \( (b_{j-1}, b_j) \). Clearly \( m < k \), and \( \dim \mathcal{F}_r < m \). In the special case where \( \tau \) is continuous on \( J \), the total number of peaks and valleys in the graph of \( \tau \) constitutes an upper bound for \( \dim \mathcal{F}_r \).

In §3 of [3] a still better bound is established for \( \dim \mathcal{F}_r \). Let \( \{b_1, b_2, \ldots, b_m\} \) be the partition defined in the previous paragraph. For each \( 1 < j < m \), define the pair

\[
\left\langle u_j, v_j \right\rangle = \left\langle \tau(b_j^-), \tau(b_j^+) \right\rangle,
\]

where \( u_j \) is regarded as \( u_j^+ \) or \( u_j^- \) depending on whether \( \tau(a_j - e) > u_j \) or \( \tau(a_j - e) < u_j \).

Two pairs \( \left\langle u_i, v_i \right\rangle \) and \( \left\langle u_j, v_j \right\rangle \) are said to be dependent if they have one or both coordinates in common. Otherwise the pairs are independent. Let \( N_r \) denote the maximal number of independent pairs. Then Theorem 2 of [3] asserts that \( \dim \mathcal{F}_r < N_r \). In this note we suggest a modified definition of dependence and present a different bound for the number of absolutely continuous measures invariant under \( \tau \).

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2. Dependence of densities. Let $\tau: J \to J$ be piecewise $C^2$ satisfying $\inf|\tau'(x)| > 1$ and let $\mathcal{D} = \{b_1, b_2, \ldots, b_m\}$ be the partition on which $\tau$ is piecewise continuous and monotonic. We shall say that $b_i$ and $b_j$ are dependent if

$$\tau(b_i - \varepsilon, b_i + \varepsilon) \cap \tau(b_j - \varepsilon, b_j + \varepsilon)$$

has positive measure for every $\varepsilon > 0$. This implies, but is not equivalent to

$$\langle \tau(b_i^-), \tau(b_i^+) \rangle \cap \langle \tau(b_j^-), \tau(b_j^+) \rangle \neq \emptyset.$$

This definition of dependence for a pair of discontinuities in $\mathcal{D}$ is reflexive, symmetric, but not transitive. A collection $\mathcal{S} \subset \mathcal{D}$ is said to be dependent if every pair of points in this collection is dependent, and maximal if $\mathcal{S}$ is not a proper subset of any dependent collection. Notice that two distinct maximal dependent collections may have nonempty intersection, and such a collection may consist of a single point. Thus, given $b_j \in \mathcal{D}$, there exists at least one and at most two maximal dependent collections containing $b_j$. In particular, when $\tau$ is continuous at $b_j$, there exists only one maximal dependent collection containing this point. Let $H_{\tau}$ be the number of distinct maximal dependent collections. Then, we have

**Theorem.** $\dim \mathcal{D}_{\tau} < H_{\tau}$.

**Proof.** We first show that if $f_1$ and $f_2$ are invariant with disjoint supports, then to each $f_i$ there corresponds one maximal dependent collection $S_i$ and $S_1 \neq S_2$. Letting $M_i = \text{spt} f_i$, it is easy to see that $\text{int} M_i$ has to contain at least one point of $\mathcal{D}$, say $b_i$. Let $S_1$ and $S_2$ be any maximal collections containing $b_1'$ and $b_2'$, respectively, and suppose $S_1 = S_2$. Then $b_1'$ and $b_2'$ are dependent. Since $\tau(M_i) \subset M_i$ a.e. [1], and $(b_1' - \varepsilon, b_1' + \varepsilon) \subset M_i$ for some $\varepsilon < 0$, the dependence of $b_1'$ and $b_2'$ implies

$$\lambda(M_1 \cap M_2) > \lambda[\tau(b_1' - \varepsilon, b_1' + \varepsilon) \cap \tau(b_2' - \varepsilon, b_2' + \varepsilon)] > 0.$$ 

This is a contradiction. Therefore, $S_1$ and $S_2$ must be distinct.

Now let $\{f_1, f_2, \ldots, f_n\}$ be a maximal set of disjoint densities invariant under $\tau$ [2]. By the preceding argument we see that there exists a 1-1 mapping from $\{f_1, \ldots, f_n\}$ into $\{S_1, \ldots, S_{H_\tau}\}$. Thus $n < H_{\tau}$. Q.E.D.

3. Examples. (a) Consider the transformation $\tau$ shown in Figure 1.

We see that $\{b_1, b_2, b_3\}$ is the unique collection which is dependent and maximal. Thus $H_{\tau} = 1$ and there exists a unique absolutely continuous measure invariant under $\tau$. The bound from [2] is 8, since there are 8 discontinuities in $\tau'$ in $(0, 1)$. 

![Figure 1](https://www.ams.org/journal-terms-of-use)
(b) Let \( \tau \) have the graph shown in Figure 2.

For each discontinuity, we give the corresponding maximal dependent collection or collections as the case may be:

- \( b_1 \): \( \{ b_1, b_3, b_5 \} \) and \( \{ b_1, b_4 \} \),
- \( b_2 \): \( \{ b_2, b_4, b_5 \} \),
- \( b_3 \): \( \{ b_1, b_2, b_5 \} \),
- \( b_4 \): \( \{ b_1, b_4 \} \) and \( \{ b_2, b_4, b_5 \} \),
- \( b_5 \): \( \{ b_1, b_3, b_5 \} \) and \( \{ b_2, b_4, b_3 \} \),
- \( b_6 \): \( \{ b_6 \} \).

There are 4 independent collections. Therefore \( \tau \) admits at most four independent invariant densities.

Notice that for this example the bound of [2] is 7, since there are 7 discontinuities of \( \tau' \) in \((0, 1)\).

REFERENCES


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