ON MEASURES OF ANALYTIC TYPE

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Abstract. The purpose of this note is to show how the characterization of an absolutely continuous or singular measure of R. Doss can be used to prove the Helson-Lowdenslager Theorem for measures of analytic type. Our methods were inspired by the work of P. Cohen and H. Davenport on the \( L^1 \) norm of exponential sums.

Let \( G \) be an infinite LCA group with character group \( \Gamma \) and \( M(G) \) the customary convolution algebra of bounded Borel measures on \( G \); let \( \hat{\cdot} \) denote the Fourier-Stieltjes transform. We say \( \Gamma \) is totally ordered if \( \Gamma \) is discrete and there exists a semigroup \( P \) such that \( (P) \cap (-P) = \{0\} \) and \( (P) \cup (-P) = \Gamma \). A measure \( \mu \in M(G) \) is said to be of analytic type if \( \hat{\mu}(\gamma) = 0 \) for all \( \gamma \in \Gamma \setminus P = \{ \gamma \in \Gamma : \gamma < 0 \} \).

For \( \mu \in M(G) \) put \( \mu = \mu_a + \mu_s \) where \( \mu_a \) is absolutely continuous with respect to Haar measure on \( G \) and \( \mu_s \) is singular with respect to Haar measure. Let \( M_a(G) \) and \( M_s(G) \) denote the space of absolutely continuous and singular measures, respectively. For any \( K \subset \Gamma \), let \( A_K \) be the set of all trigonometric polynomials on \( G \) of the form \( p(x) = \sum c_\gamma \gamma(-x) \) such that \( \| p \|_\infty < 1 \) and \( \gamma \notin K \). For \( \mu \in M(G) \) we define

\[
\| \mu \|_K = \sup_{p \in A_K} |\sum c_\gamma \hat{\mu}(\gamma)| \quad \text{and} \quad \lim_{K \to \infty} \| \mu \|_K = \inf_K \| \mu \|_K ;
\]

here \( K \) varies through some presumably interesting family of subsets of \( \Gamma \); for example \( K \) can run through the compact sets.

The following quantitatively precise result can be found in [4].

Proposition 1 (R. Doss). For \( \mu \in M(G) \) and \( K \) compact in \( \Gamma \)

\[
\lim_{K \to \infty} \| \mu \|_K = \| \mu_a \| .
\]

A modicum of complex variables will be needed in the sequel.

Lemma 1. Let \( T(z) = z/(z + 3a) \) where \( 0 < a < 1 \). For \( M \in \mathbb{Z}^+ \) put \( S_M = \{ z \in \mathbb{C} : -a < \Re z < 1, -M < \Im z < M \} \). Then given \( \delta > 0 \) and \( M \in \mathbb{Z}^+ \) there is a polynomial of the form

\[
T_M(z) = a_1z + \cdots + a_nz^n \quad (\#)
\]
such that

(i) \(|T(z) - T_M(z)| < \delta\) for all \(z \in S_M\),
(ii) \(\text{Re}\{T_M(z)\} > \{1/(1 + 3a)\}\text{Re} z - \delta/2\) if \(0 < \text{Re}\ z < 1\) and
(iii) \(|T_M(z)|^2 < 1 + \delta\) for all \(z \in S_M\).

**Proof.** Notice that \(\text{Re}\{T(z)\} > \{1/(1 + 3a)\}\text{Re} z\) if \(0 < \text{Re} z < 1\) and \(T\) maps \(S_M\) into the open unit disc. Fix any real number \(r\) greater than \(\frac{1 + M^2}{4a}\). Then the disc \(\{z \in \mathbb{C}: |z - r| < 3a + r\}\) contains \(S_M\). Using the fact that \(T(0) = 0\), we expand \(T\) in a power series about \(r\) to obtain a polynomial of the form \((\#)\) satisfying (i)–(iii).

**Lemma 2.** Let \(T(z) = z/(z + 3a)\) \((0 < a < 1)\). Then given \(\varepsilon > 0\) there is a \(\xi > 0\) and a \(b > 0\) such that if \(- A < \text{Re} \ z < 1\) and \(\text{Re} \left\{z/(z + 3a)\right\} < \xi\) then \(|T(z)| < \varepsilon/4\) and if \(c = -b/(-b + 3a)\) then \(-c/(-c + \xi) < \varepsilon\).

**Theorem 1.** Let \(\Gamma\) be totally ordered, \(v \in M_c(\Gamma)\) and \(|v|\) concentrated on \(\Omega\). Then given \(\varepsilon > 0\) there is a trigonometric polynomial \(k\) having only positive frequencies, \(E \subset \Omega\) and \(F \subset G\) such that

\[
(a) \|k\|_\infty < 1 + \varepsilon, \\
(b) |k - 1| < \varepsilon \text{ on } \Omega \setminus E \text{ and } |v|(E) < \varepsilon, \\
(c) |k| < \varepsilon \text{ on } G \setminus F \text{ and } m_G(F) < \varepsilon.
\]

Here \(m_G\) is the normalized Haar measure on \(G\).

**Proof.** Let \(v \in M_c(\Gamma)\) and \(|v|\) concentrated on \(\Omega\); assume \(||v|| = 1\). Let \(0 < \varepsilon < 1\) be given; choose \(\delta > 0\) such that \((4\delta + \delta^2)^{1/2} < \varepsilon\). Let \(a > 0\), \(a < \delta/6\); put \(T(z) = z/(z + 3a)\). For \(a > 0\) and \(\varepsilon > 0\) we see via Lemma 2 that there is a \(\xi > 0\) and a \(b > 0\) satisfying

\[
b < a, \\
-\frac{c}{c + \xi} < \varepsilon \quad \text{where } c = \frac{-b}{-b + 3a},
\]

\[
\text{if } -b < \text{Re} \ z < 1 \text{ and } \text{Re} \left\{T(z)\right\} < \xi \text{ then } |T(z)| < \varepsilon/4.
\]

Put \(0 < \beta < (\delta/Q)^2\) where \(1/(Q - 1) < b\) and \(Q \in \mathbb{Z}^+, \ Q > 4\).

It follows from Proposition 1 that we can inductively choose \(Q\) trigonometric polynomials \(p_1, \ldots, p_Q\) such that

\[
\|p_i\|_\infty < 1 \quad (i = 1, \ldots, Q),
\]

\[
\int_G p_i \ d|v| > 1 - \beta^2 \quad (i = 1, \ldots, Q)
\]

and

\[
\tilde{p}_s p_t \text{ has frequencies in } \Gamma \setminus \{0\} \text{ for } s < t.
\]

For each trigonometric polynomial \(p_i\) define \(E_i = \{x \in \Omega: \text{Re} \ p_i(x) < 1 - \beta\}\). An easy calculation based on (1.5) gives \(|v|(E_i) < \beta\) for each \(i\). Put \(E = \bigcup_i^Q E_i\); then \(|v|(E) < \delta\).
As in [2], we now define the trigonometric polynomial \( g \) by
\[
g = \frac{2}{Q^2 - Q} \sum_{-2 < t} \tilde{\beta}_t p_t = u + i\tilde{v}.
\]
An easy computation using (1.4) shows that \( \|g\|_\infty < 1 \) and \( u > -1/(Q - 1) \); thus \( -b < u < 1 \). A direct computation also establishes
\[
u(x) > 1 - 4\beta \quad (x \in \Omega \setminus E). \tag{1.7}
\]
Write \( g = g^+ + g^- \) where \( g^+ \) is the analytic projection of \( g \). Put \( h = (g^+) + (g^-) \); observe that \( h \) has only positive frequencies. Unfortunately, although \( \|g\|_\infty < 1 \), we cannot say as much of \( h \). However, we will make much of the fact that \( h \) and \( g \) have the same real part.

Let \( h = u + iv \); there is an \( M \in \mathbb{Z}^+ \) such that \( -M < v < M \) and thus by Lemma 1 \( T_M(h) = k \) is a trigonometric polynomial with positive frequencies such that \( |T(h) - k| < \epsilon/4 \),
\[
||k||_\infty^2 < 1 + \delta \tag{1.8}
\]
and via (1.7)
\[
\text{Re}(k(x)) > 1 - \delta \quad \text{for all} \ x \in \Omega \setminus E. \tag{1.9}
\]
It is a direct consequence of (1.8) and (1.9) that \( |1 - k| < \epsilon \) on \( \Omega \setminus E \). Thus, (a) and (b) are now established.

We now wish to confirm (c). Put \( k = u_1 + i\tilde{v}_1 \) and \( F = \{ x \in G : u_1 > \xi \} \). Inasmuch as \( k \) has only positive frequencies we see that \( \int_G u_1(\theta) \ d\theta = 0 \). However, \( c < u_1 \) and \( \int_G u_1(\theta) \ d\theta = 0 \) combine with (1.2) to give \( m_G(F) < \epsilon \). It follows from (1.3) and \( |T(h) - k| < \epsilon/4 \) that \( |k| < \epsilon \) on \( \Omega \setminus F \). Our proof is complete.

**Corollary 1 (Helson-Lowdenslager [6]).** If \( \Gamma \) is totally ordered and \( \mu \in \mathcal{M}(G) \) is of analytic type so is \( \mu_\gamma \) and \( \hat{\mu}_\gamma(0) = 0 \).

**Proof.** Let \( \mu \in \mathcal{M}(G) \) be of analytic type; put \( \mu = \mu_\gamma + \mu_\gamma; \) we gather from Theorem 1 that there is a sequence \( \langle k_n \rangle_\infty \) of trigonometric polynomials having only positive frequencies such that \( k_n \to 1 \) a.e. mod \( \mu_\gamma \), \( k_n \to 0 \) a.e. mod \( m_G \) and \( \|k_n\|_\infty < 2 \). Notice that for all \( \gamma \in \Gamma \)
\[
\hat{\mu}_\gamma(\gamma) = \int_G (1 - k_n(x))\gamma(-x) \ d\mu_\gamma + \int_G k_n(x)\gamma(-x) \ d\mu_\gamma.
\]
Fix \( \gamma_0 < 0 \); it follows that
\[
|\hat{\mu}_\gamma(\gamma_0)| \leq \int_G |1 - k_n| \ d|\mu_\gamma| + \int_G k_n(x)\gamma_0(-x) \ d(\mu - \mu_\gamma);
\]
since \( \mu \) is of analytic type, \( k_n \) has only positive frequencies, and \( \gamma_0 < 0 \), we get
\[
|\hat{\mu}_\gamma(\gamma_0)| \leq \int_G |1 - k_n| \ d|\mu_\gamma| + \int_G |k_n| \ d|\mu_\gamma|.
\]
Well, let \( n \to \infty \) in the above inequality; then both summands on the right side of the inequality converge to zero by the Lebesgue Dominated Convergence Theorem. Our proof is complete.
It is not hard to see that if $\Gamma_0 \subset \Gamma$ has finite index in $\Gamma$, we can also require in Theorem 1 that $k$ have frequencies in $\Gamma_0 \setminus \{0\}$. With this remark in mind we obtain an infinite-dimensional version of a result of S. Bochner [1].

**Corollary 2.** For any cardinal number $\omega$ let $T^\omega$ denote the unrestricted product of $\omega$ copies of $T$. Put $Q^\omega = \{\langle x_\alpha \rangle: x_\alpha \in \mathbb{Z}, x_\alpha > 0 \text{ for all } \alpha \} \subset \hat{T}^\omega$. If $\mu \in M(T^\omega)$ and $\text{supp } \mu \subset Q^\omega$ then $\mu \in M_0(T^\omega)$.

**Proof.** Suppose $\mu \in M(T^\omega)$ and $\text{supp } \mu \subset Q^\omega$; fix $\xi_\alpha = \langle x_\alpha \rangle \in \hat{T}^\omega$ and choose $N > |\xi_\alpha|$ for all $\alpha (N \in \mathbb{Z}^+)$. Let $\Gamma_0^\omega$ be the group generated by $\xi_\alpha$; there are a finite number of fixed indices $\alpha_1, \ldots, \alpha_j$ such that if $\langle x_\alpha \rangle \in \Gamma_0^\omega$ then $x_\alpha = 0$ for all $\alpha \not\in \{\alpha_1, \ldots, \alpha_j\}$. Let $\Gamma_0$ be the subgroup of $\hat{T}^\omega$ such that if $\langle x_\alpha \rangle \in \Gamma_0$ then $x_\alpha \in \mathbb{N} \mathbb{Z}$ for $i = 1, \ldots, j$.

Well-order the indexing set for $T^\omega$ and put $P^+$ equal to the positive cone (with respect to the usual lexicographic order) induced by the well-ordering. Since $\Gamma_0$ has finite index in $\hat{T}^\omega$, there is a sequence of trigonometric polynomials $\langle k_n \rangle_1^\infty$ having frequencies in $P^+ \cap \Gamma_0$, $\|k_n\|_\infty < 2$, such that $k_n \to 1$ a.e. mod $|\mu|$ and $k_n \to 0$ a.e. mod $m_{T^\omega}$. Thus

$$\left| \hat{\mu}_i(\xi_0) \right| < \int_{T^\omega} |1 - k_n| \, d|\mu_i| + \left| \int_{T^\omega} k_n(x)\xi_0(-x) \, d(\mu - \mu_0) \right|,$$

since supp $\mu \subset Q^\omega$ and Freq $k_n \subset P^+ \cap \Gamma_0$, we obtain

$$\left| \hat{\mu}_i(\xi_0) \right| < \int_{T^\omega} |1 - k_n| \, d|\mu_i| + \left| \int_{T^\omega} k_n(x)\xi_0(-x) \, d\mu_0 \right|< \int_{T^\omega} |1 - k_n| \, d|\mu_i| + \int_{T^\omega} |k_n(x)| \, d|\mu_0|.$$

Letting $n \to \infty$ yields $\hat{\mu}_i(\xi_0) = 0$. Our proof is complete.

Perhaps Corollary 2 is new; in any event see [3, p. 191]. Theorem 1 can be placed in perspective by a reading of [5, pp. 43-44].

For discrete groups $\Gamma$, there is available another notion of order: If there exists a nontrivial homomorphism $\phi: \Gamma \to \mathbb{R}$, where $\mathbb{R}$ is the additive group of reals, we say $\Gamma$ is $\phi$-ordered and put $P = \phi^{-1}([0, \infty))$. Also let $P_\phi = \phi^{-1}([N, \infty))$; $\gamma$ is $\phi$-positive ($\phi$-negative) if $\phi(\gamma) > 0$ ($\phi(\gamma) < 0$). Again, $\mu \in M(G)$ is said to be of analytic type if $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus P$. We point out that the obvious analogue of Theorem 1 is valid with the same proof if the kernel of $\phi$ is finite. Repeating the proof of Corollary 1 we obtain the following result: If ker $\phi$ is finite and $\mu \in M(G)$ is of analytic type so is $\mu$, and $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \ker \phi$.

We say a subset $K \subset \Gamma$ is $\phi$-finite if there exists a finite set $B \subset \mathbb{R}$ such that $\phi(K) \subset B$. A subset $K \subset \Gamma$ is $\phi$-bounded if there exists a bounded set $B \subset \mathbb{R}$ such that $\phi(K) \subset B$.

We denote $(\ker \phi)^\perp$ by $G_0$. $M_{\phi G_0}(G)$ denotes the smallest $L$-ideal in $M(G)$ containing all measures of the form $m_{G_0} \ast \rho$ where $m_{G_0}$ is Haar measure on $G_0$ and $\rho \in M(G)$. Put $M_{\phi G_0}(G) = M_{\phi G_0}(G)^\perp = \{\nu \in M(G): \nu \perp \mu \text{ for each } \mu \in M_{\phi G_0}(G)\}$. For $\mu \in M(G)$ put $\mu = \mu_{\phi G_0} + \mu_{\phi G_0}$ where $\mu_{\phi G_0} \in M_{\phi G_0}(G)$ and $\mu_{\phi G_0} \in M_{\phi G_0}(G)$. 


\( \phi \) induces a continuous homomorphism \( \phi^*: R \rightarrow G \) given by \( \langle \phi^*(x), \gamma \rangle = e^{-i(x \phi(\gamma))} \) \( (x \in R, \gamma \in \Gamma) \). Let \( \Psi: M(R) \rightarrow M(G) \) where \( \Psi(\mu)(E) = \mu(\phi^{-1}(E)) \) and \( E \) is Borel in \( G \). \( M_{\text{ap}}(G) \) denotes the smallest \( L \)-ideal in \( M(G) \) containing all measures of the form \( \Psi(\omega) \ast \rho \) where \( \omega \in M_\omega(R) \) and \( \rho \in M(G) \). Put \( M_{\text{ap}}(G) = M_{\text{ap}}(G)^1 = \{ \nu \in M(G): \nu \perp \mu \text{ for each } \mu \in M_{\text{ap}}(G) \} \). For \( \mu \in M(G) \) put \( \mu = \mu_{\text{ap}} + \mu_{\text{ap}} \) where \( \mu_{\text{ap}} \in M_{\text{ap}}(G) \) and \( \mu_{\text{ap}} \in M_{\text{ap}}(G) \). It is easy to see that \( M_{\text{ap}}(G) \) coincides with the measures \( \mu \in M(G) \) that translate continuously in the direction of \( \phi \); i.e., \( \lim_{x \to 0} \| \phi^*(x) \ast \mu - \mu \| = 0 \).

K. de Leeuw and I. Glicksberg proved that if \( \Gamma \) is \( \phi \)-ordered and \( \mu \in M(G) \) is of analytic type then \( \mu \in M_{\text{ap}}(G) \); see [3]. An easy consequence of the de Leeuw-Glicksberg result is: If \( \mu \in M(G) \) is of analytic type and \( \Gamma \) is \( \phi \)-ordered then \( \mu_{\text{ap}} \) is also of analytic type; see [3, p. 186]. We now show how to obtain these last two results with the methods of this paper.

**Proposition 2.** For \( \mu \in M(G) \) and \( K \phi \)-finite in \( \Gamma \)

\[
\lim_{K \to \infty} \| \mu \|_K = \| \mu_{\text{ap}} \|.
\]

For \( \mu \in M(G) \) and \( K \phi \)-bounded in \( \Gamma \)

\[
\lim_{K \to \infty} \| \mu \|_K = \| \mu_{\text{ap}} \|.
\]

**Proof.** The proof follows from easy modifications of [4]; see also [7].

**Theorem 2.** Let \( \Gamma \) be \( \phi \)-ordered. Let \( \nu \in M_{\delta G_0}(G) \) and \( \lambda \in M_{\delta G_0}(G) \), or let \( \nu \in M_{\text{ap}}(G) \) and \( \lambda \in M_{\text{ap}}(G) \). Let \( \nu \) be concentrated on \( \Omega_\nu \) and \( \lambda \) be concentrated on \( \Omega_\lambda \). Then given \( \epsilon > 0 \) there is a trigonometric polynomial \( k \) having only \( \phi \)-positive frequencies, \( E \subset \Omega_\nu, F \subset \Omega_\lambda \) such that

(a) \( \| k \|_\infty < 1 + \epsilon \);

(b) \( | k - 1 | < \epsilon \) on \( \Omega_\nu \setminus E \) and \( | \nu(E) | < \epsilon \);

(c) \( | k | < \epsilon \) on \( \Omega_\lambda \setminus F \) and \( | \lambda(F) | < \epsilon \).

Furthermore, in the case \( \nu \in M_{\text{ap}}(G) \) and \( \lambda \in M_{\text{ap}}(G) \) we may choose \( k \) to have frequencies in \( P_N \) for any choice of \( N \).

**Proof.** The proof is to interpret the argument of Theorem 1 in the \( \phi \)-ordered context.

First, suppose \( \nu \in M_{\delta G_0}(G) \) and \( \lambda \in M_{\delta G_0}(G) \). We may take \( \lambda \) to be nonnegative. \( \lambda \ll m_{G_0} \ast \lambda \). Thus suppose \( \hat{\lambda} \) vanishes off ker \( \phi \) and in addition that \( \| \lambda \| = 1 \). In the choice of the trigonometric polynomials \( p_1, \ldots, p_Q \) replace (1.6) by

\[
\tilde{p}_s p_t \text{ has frequencies in } \Gamma \setminus \ker \phi \text{ for } s < t. \tag{2.1}
\]

This can be done by Proposition 2. The remainder of the argument is repeated with \( m_G \) replaced by \( \lambda \). The crucial observation is that

\[
\int_G u_1(\theta) \, d\lambda(\theta) = 0
\]

since \( k \) has no frequencies in ker \( \phi \) and supp \( \hat{\lambda} \subset \ker \phi \).

Now suppose \( \nu \in M_{\text{ap}}(G) \) and \( \lambda \in M_{\text{ap}}(G) \). Take \( \lambda \) to be nonnegative. Let \( f \in M(R), f > 0 \), have compactly supported transform. Then \( \lambda \ll \Psi(f) \ast \lambda \) and we
may thus suppose \( \hat{\lambda} \) has \( \phi \)-bounded support. Put \( \|\lambda\| = 1 \) and fix \( N \in \mathbb{Z}^+ \) with 
\( \text{supp} \hat{\lambda} \subset \Gamma \setminus P_N \). In the choice of the trigonometric polynomials \( p_1, \ldots, p_Q \) replace (1.6) by
\[
\hat{p}_s p_t \text{ has frequencies in } P_N \cup (-P_N) \text{ for } s < t.
\] (2.2)
This can be done by Proposition 2. The remainder of the argument is repeated with \( m_G \) replaced by \( \lambda \). The crucial observation is that
\[
\int G u_t(\theta) \, d\lambda(\theta) = 0
\]
since \( k \) has its frequencies in \( P_N \).

**Corollary 3.** Let \( \mu \in M(G) \) be of analytic type for \( \Gamma \) having a \( \phi \)-ordering. Then

1. \( \mu_{\phi_\gamma} = 0 \),
2. \( \hat{\mu}_{G,\phi_\gamma}(\gamma) = 0 \) for all \( \gamma \) with \( \phi(\gamma) < 0 \),
3. \( \hat{\mu}_\gamma(\gamma) = 0 \) for all \( \gamma \) with \( \phi(\gamma) < 0 \).

**Proof.** Put \( \mu = \mu_{\phi_\gamma} + \mu_\phi \) and let \( \gamma_0 \in \Gamma \). Select from Theorem 2 a sequence \( \langle k_n \rangle \) of trigonometric polynomials having frequencies in \( P_N \) where \( |\phi(\gamma_0)| < N \) such that \( k_n \to 1 \) a.e. \( \mod |\mu_{\phi_\gamma}| \) and \( k_n \to 0 \) a.e. \( \mod |\mu_\phi| \). Then as in the proof of Corollary 1
\[
|\hat{\mu}_{\phi_\gamma}(\gamma_0)| \leq \int G |1 - k_n| \, d|\mu_{\phi_\gamma}| + \left| \int G k_n(x) \gamma_0(-x) \, d(\mu - \mu_{\phi_\gamma}) \right|.
\]
But \( k_n \gamma_0 \) has only \( \phi \)-positive frequencies since \( \text{Freq } k_n \subset P_N \). So
\[
|\hat{\mu}_\phi(\gamma_0)| \leq \int G |1 - k_n| \, d|\mu_\phi| + \int G |k_n| \, d|\mu_{\phi_\gamma}|.
\]
Both summands on the right converge to 0 as \( n \to \infty \).

Now put \( \mu = (\mu_{\phi_\gamma} - \mu_\phi) = \mu_{G,\phi} + \mu_{G,\phi_\gamma} \). From Theorem 2 select a sequence \( \langle k_n \rangle \) of trigonometric polynomials having \( \phi \)-positive frequencies such that \( k_n \to 1 \) a.e. \( \mod |\mu_{G,\phi}| \) and \( k_n \to 0 \) a.e. \( \mod |\mu_{G,\phi_\gamma}| \) and \( \|k_n\|_\infty < 2 \). Let \( \gamma_0 \in \{ \gamma : \phi(\gamma) < 0 \} \). Then \( \gamma_0 k_n \) has only \( \phi \)-positive frequencies and the argument of the preceding paragraph repeats to give
\[
\hat{\mu}_{G,\phi}(\gamma_0) = 0 \quad \text{on } \{ \gamma : \phi(\gamma) < 0 \}.
\]

We now have \( \mu_{G,\phi} \) is analytic. Write \( \mu_{G,\phi} = \mu_\phi + \mu_{G,\phi, s} \) where \( \mu_{G,\phi, s} \) is the singular part. It remains to see \( \mu_{G,\phi, s} \) is analytic. But \( \mu_{G,\phi, s} \ll m_{G,0} \cdot |\mu_{G,\phi, s}| = \nu \) is a positive singular measure, and therefore by Proposition 1 we may choose for any given finite set \( F \subset \Gamma_0 \) a sequence of trigonometric polynomials \( \langle k_n \rangle \) such that 
\( \text{Freq } k_n \subset \Gamma_0 \setminus F, k_n \to 1 \) a.e. \( \mod |\mu_{G,\phi, s}| \) and \( \|k_n\|_\infty < 2 \). Choose a trigonometric polynomial \( s \) so that \( \|s - \mu_\phi\| < \epsilon \). Let \( \gamma_0 \) be \( \phi \)-negative. Determine \( \langle k_n \rangle \) by the choice \( F = \gamma_0(\text{Freq } s) \cap \Gamma_0 \). Now
\[
\hat{\mu}_{\phi, s}(\gamma_0) = \int \gamma_0(1 - k_n) \, d\mu_{\phi, s} + \int \gamma_0 k_n \, d\mu_{\phi, s, s} \\
= \int \gamma_0(1 - k_n) \, d\mu_{\phi, s} - \int \gamma_0 k_n \, d\mu_\phi \quad \text{(by the analyticity of } \mu_{\phi_\gamma}).}
So $|\hat{\phi}_n, s(\gamma_0)| < \int |1 - k_n| d|\mu_{\phi_n, s}| + |\int \tilde{\gamma}_0 k_n d\mu_n|$. $\int |1 - k_n| d|\mu_{\phi_n, s}| \to 0$ as $n \to \infty$. And $|\int \tilde{\gamma}_0 k_n d\mu_n| < |\int \tilde{\gamma}_0 k_n s d\theta| + 2\varepsilon = 2\varepsilon$ (since $\gamma_0 k_n \in \infty < 2$).

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