PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

ON MEASURES OF ANALYTIC TYPE

LOUIS PIGNO AND BRENT SMITH

ABSTRACT. The purpose of this note is to show how the characterization of an absolutely continuous or singular measure of R. Doss can be used to prove the Helson-Lowdenslager Theorem for measures of analytic type. Our methods were inspired by the work of P. Cohen and H. Davenport on the L^1 norm of exponential sums.

Let G be an infinite LCA group with character group Γ and $M(G)$ the customary convolution algebra of bounded Borel measures on G; let $\hat{}$ denote the Fourier-Stieltjes transform. We say Γ is *totally ordered* if Γ is discrete and there exists a semigroup P such that $(P) \cap (-P) = \{0\}$ and $(P) \cup (-P) = \Gamma$. A measure $\mu \in M(G)$ is said to be of *analytic type* if $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus P = \{\gamma \in \Gamma: \gamma < 0\}$.

For $\mu \in M(G)$ put $\mu = \mu_a + \mu_s$ where μ_a is absolutely continuous with respect to Haar measure on G and μ_s is singular with respect to Haar measure. Let $M_a(G)$ and $M_s(G)$ denote the space of absolutely continuous and singular measures, respectively. For any $K \subset \Gamma$, let A_K be the set of all trigonometric polynomials on G of the form $p(x) = \sum c_\gamma \gamma(-x)$ such that $\|p\|_\infty \leqslant 1$ and $\gamma \notin K$. For $\mu \in M(G)$ we define

$$\|\mu\|_K = \sup_{p \in A_K} |\sum c_\gamma \hat{\mu}(\gamma)| \quad \text{and} \quad \lim_{K \to \infty} \|\mu\|_K = \inf_K \|\mu\|_K;$$

here K varies through some presumably interesting family of subsets of Γ; for example K can run through the compact sets.

The following quantitatively precise result can be found in [4].

PROPOSITION 1 (R. DOSS). *For $\mu \in M(G)$ and K compact in Γ*

$$\lim_{K \to \infty} \|\mu\|_K = \|\mu_s\|.$$

A modicum of complex variables will be needed in the sequel.

LEMMA 1. *Let $T(z) = z/(z + 3a)$ where $0 < a < 1$. For $M \in \mathbf{Z}^+$ put $S_M = \{z \in \mathbf{C}: -a \leqslant \operatorname{Re} z \leqslant 1, -M \leqslant \operatorname{Im} z \leqslant M\}$. Then given $\delta > 0$ and $M \in \mathbf{Z}^+$ there is a polynomial of the form*

$$T_M(z) = a_1 z + \cdots + a_n z^n \tag{\#}$$

Received by the editors May 6, 1980.
1980 *Mathematics Subject Classification*. Primary 43A17.

such that

(i) $|T(z) - T_M(z)| < \delta$ *for all* $z \in S_M$,

(ii) $\mathrm{Re}\{T_M(z)\} \geqslant \{1/(1+3a)\}\mathrm{Re}\, z - \delta/2$ *if* $0 \leqslant \mathrm{Re}\, z \leqslant 1$ *and*

(iii) $|T_M(z)|^2 \leqslant 1 + \delta$ *for all* $z \in S_M$.

PROOF. Notice that $\mathrm{Re}[T(z)] \geqslant \{1/(1+3a)\}\mathrm{Re}\, z$ if $0 \leqslant \mathrm{Re}\, z \leqslant 1$ and T maps S_M into the open unit disc. Fix any real number r greater than $[1 + M^2]/4a$. Then the disc $\{z \in \mathbf{C}: |z - r| \leqslant 3a + r\}$ contains S_M. Using the fact that $T(0) = 0$, we expand T in a power series about r to obtain a polynomial of the form (#) satisfying (i)–(iii).

LEMMA 2. *Let* $T(z) = z/(z+3a)$ $(0 < a < 1)$. *Then given* $\varepsilon > 0$ *there is a* $\xi > 0$ *and a* $b > 0$ *such that if* $-b \leqslant \mathrm{Re}\, z \leqslant 1$ *and* $\mathrm{Re}\,\{z/(z+3a)\} \leqslant \xi$ *then* $|T(z)| < \varepsilon/4$ *and if* $c = -b/(-b+3a)$ *then* $-c/(-c+\xi) < \varepsilon$.

THEOREM 1. *Let* Γ *be totally ordered,* $\nu \in M_s(G)$ *and* $|\nu|$ *concentrated on* Ω. *Then given* $\varepsilon > 0$ *there is a trigonometric polynomial* k *having only positive frequencies,* $E \subset \Omega$ *and* $F \subset G$ *such that*

(a) $\|k\|_\infty < 1 + \varepsilon$,

(b) $|k - 1| < \varepsilon$ *on* $\Omega \setminus E$ *and* $|\nu|(E) < \varepsilon$,

(c) $|k| < \varepsilon$ *on* $G \setminus F$ *and* $m_G(F) < \varepsilon$.

Here m_G is the normalized Haar measure on G.

PROOF. Let $\nu \in M_s(G)$ and $|\nu|$ concentrated on Ω; assume $\|\nu\| = 1$. Let $0 < \varepsilon < 1$ be given; choose $\delta > 0$ such that $(4\delta + \delta^2)^{1/2} < \varepsilon$. Let $a > 0$, $a < \delta/6$; put $T(z) = z/(z+3a)$. For $a > 0$ and $\varepsilon > 0$ we see via Lemma 2 that there is a $\xi > 0$ and a $b > 0$ satisfying

$$b < a, \tag{1.1}$$

$$\frac{-c}{-c+\xi} < \varepsilon \quad \text{where } c = \frac{-b}{-b+3a}, \tag{1.2}$$

$$\text{if } -b \leqslant \mathrm{Re}\, z \leqslant 1 \text{ and } \mathrm{Re}\,\{T(z)\} \leqslant \xi \text{ then } |T(z)| \leqslant \varepsilon/4. \tag{1.3}$$

Put $0 < \beta < (\delta/Q)^2$ where $1/(Q-1) < b$ and $Q \in \mathbf{Z}^+$, $Q \geqslant 4$.

It follows from Proposition 1 that we can inductively choose Q trigonometric polynomials $p_1, \ldots, p_Q$ such that

$$\|p_i\|_\infty \leqslant 1 \qquad (i = 1, \ldots, Q), \tag{1.4}$$

$$\int_G p_i \, d|\nu| > 1 - \beta^2 \qquad (i = 1, \ldots, Q) \tag{1.5}$$

and

$$\bar{p}_s p_t \text{ has frequencies in } \Gamma \setminus \{0\} \text{ for } s < t. \tag{1.6}$$

For each trigonometric polynomial p_i define $E_i = \{x \in \Omega: \mathrm{Re}\, p_i(x) \leqslant 1 - \beta\}$. An easy calculation based on (1.5) gives $|\nu|(E_i) \leqslant \beta$ for each i. Put $E = \bigcup_1^Q E_i$; then $|\nu|(E) < \delta$.

As in [2], we now define the trigonometric polynomial g by

$$g = \frac{2}{Q^2 - Q} \sum_{s<t} \bar{p}_s p_t = u + i\tilde{v}.$$

An easy computation using (1.4) shows that $\|g\|_\infty \leqslant 1$ and $u \geqslant -1/(Q-1)$; thus $-b \leqslant u \leqslant 1$. A direct computation also establishes

$$u(x) > 1 - 4\beta \qquad (x \in \Omega \setminus E). \tag{1.7}$$

Write $g = g^+ + g^-$ where g^+ is the analytic projection of g. Put $h = (g^+) + \overline{(g^-)}$; observe that h has only positive frequencies. Unfortunately, although $\|g\|_\infty \leqslant 1$, we cannot say as much of h. However, we will make much of the fact that h and g have the same real part.

Let $h = u + iv$; there is an $M \in \mathbf{Z}^+$ such that $-M \leqslant v \leqslant M$ and thus by Lemma 1 $T_M(h) = k$ is a trigonometric polynomial with positive frequencies such that $|T(h) - k| < \varepsilon/4$,

$$\|k\|_\infty^2 \leqslant 1 + \delta \tag{1.8}$$

and via (1.7)

$$\operatorname{Re} k(x) > 1 - \delta \quad \text{for all } x \in \Omega \setminus E. \tag{1.9}$$

It is a direct consequence of (1.8) and (1.9) that $|1 - k| < \varepsilon$ on $\Omega \setminus E$. Thus, (a) and (b) are now established.

We now wish to confirm (c). Put $k = u_1 + iv_1$ and $F = \{x \in G\colon u_1 > \xi\}$. Inasmuch as k has only positive frequencies we see that $\int_G u_1(\theta)\, d\theta = 0$. However, $c \leqslant u_1$ and $\int_G u_1(\theta)\, d\theta = 0$ combine with (1.2) to give $m_G(F) < \varepsilon$. It follows from (1.3) and $|T(h) - k| \leqslant \varepsilon/4$ that $|k| < \varepsilon$ on $\Omega \setminus F$. Our proof is complete.

COROLLARY 1 (HELSON-LOWDENSLAGER [6]). *If Γ is totally ordered and $\mu \in M(G)$ is of analytic type so is μ_s and $\hat{\mu}_s(0) = 0$.*

PROOF. Let $\mu \in M(G)$ be of analytic type; put $\mu = \mu_a + \mu_s$; we gather from Theorem 1 that there is a sequence $\langle k_n \rangle_1^\infty$ of trigonometric polynomials having only positive frequencies such that $k_n \to 1$ a.e. mod $|\mu_s|$, $k_n \to 0$ a.e. mod m_G and $\|k_n\|_\infty \leqslant 2$. Notice that for all $\gamma \in \Gamma$

$$\hat{\mu}_s(\gamma) = \int_G (1 - k_n(x))\gamma(-x)\, d\mu_s + \int_G k_n(x)\gamma(-x)\, d\mu_s.$$

Fix $\gamma_0 \leqslant 0$; it follows that

$$|\hat{\mu}_s(\gamma_0)| \leqslant \int_G |1 - k_n|\, d|\mu_s| + \left| \int_G k_n(x)\gamma_0(-x)\, d(\mu - \mu_a) \right|;$$

since μ is of analytic type, k_n has only positive frequencies, and $\gamma_0 \leqslant 0$, we get

$$|\hat{\mu}_s(\gamma_0)| \leqslant \int_G |1 - k_n|\, d|\mu_s| + \int_G |k_n|\, d|\mu_a|.$$

Well, let $n \to \infty$ in the above inequality; then both summands on the right side of the inequality converge to zero by the Lebesgue Dominated Convergence Theorem. Our proof is complete.

It is not hard to see that if $\Gamma_0 \subset \Gamma$ has finite index in Γ, we can also require in Theorem 1 that k have frequencies in $\Gamma_0 \setminus \{0\}$. With this remark in mind we obtain an infinite-dimensional version of a result of S. Bochner [1].

COROLLARY 2. *For any cardinal number ω let $\mathbf{T}^\omega$ denote the unrestricted product of ω copies of* $\mathbf{T}$. *Put $Q^\omega = \{\langle x_\alpha \rangle : x_\alpha \in \mathbf{Z}, x_\alpha \geqslant 0$ for all $\alpha\} \subset \hat{\mathbf{T}}^\omega$. If $\mu \in M(\mathbf{T}^\omega)$ and* supp $\hat{\mu} \subset Q^\omega$ *then $\mu \in M_a(\mathbf{T}^\omega)$.*

PROOF. Suppose $\mu \in M(\mathbf{T}^\omega)$ and supp $\hat{\mu} \subset Q^\omega$; fix $\xi_0 = \langle \xi_\alpha \rangle \in \hat{\mathbf{T}}^\omega$ and choose $N > |\xi_\alpha|$ for all α ($N \in \mathbf{Z}^+$). Let Γ_0^* be the group generated by ξ_0; there are a finite number of fixed indices $\alpha_1, \ldots, \alpha_j$ such that if $\langle y_\alpha \rangle \in \Gamma_0^*$ then $y_\alpha = 0$ for all $\alpha \notin \{\alpha_1, \ldots, \alpha_j\}$. Let Γ_0 be the subgroup of $\hat{\mathbf{T}}^\omega$ such that if $\langle w_\alpha \rangle \in \Gamma_0$ then $w_{\alpha_i} \in N\mathbf{Z}$ for $i = 1, \ldots, j$.

Well-order the indexing set for $\hat{\mathbf{T}}^\omega$ and put P^+ equal to the positive cone (with respect to the usual lexicographic order) induced by the well-ordering. Since Γ_0 has finite index in $\hat{\mathbf{T}}^\omega$, there is a sequence of trigonometric polynomials $\langle k_n \rangle_1^\infty$ having frequencies in $P^+ \cap \Gamma_0$, $\|k_n\|_\infty \leqslant 2$, such that $k_n \to 1$ a.e. mod $|\mu_s|$ and $k_n \to 0$ a.e. mod $m_{\mathbf{T}^\omega}$. Thus

$$|\hat{\mu}_s(\xi_0)| \leqslant \int_{\mathbf{T}^\omega} |1 - k_n| \, d|\mu_s| + \left| \int_{\mathbf{T}^\omega} k_n(x)\xi_0(-x) \, d(\mu - \mu_a) \right|;$$

since supp $\hat{\mu} \subset Q^\omega$ and Freq $k_n \subset P^+ \cap \Gamma_0$, we obtain

$$\begin{aligned} |\hat{\mu}_s(\xi_0)| &\leqslant \int_{\mathbf{T}^\omega} |1 - k_n| \, d|\mu_s| + \left| \int_{\mathbf{T}^\omega} k_n(x)\xi_0(-x) \, d\mu_a \right| \\ &\leqslant \int_{\mathbf{T}^\omega} |1 - k_n| \, d|\mu_s| + \int_{\mathbf{T}^\omega} |k_n(x)| \, d|\mu_a|. \end{aligned}$$

Letting $n \to \infty$ yields $\hat{\mu}_s(\xi_0) = 0$. Our proof is complete.

Perhaps Corollary 2 is new; in any event see [3, p. 191]. Theorem 1 can be placed in perspective by a reading of [5, pp. 43–44].

For discrete groups Γ, there is available another notion of order: If there exists a nontrivial homomorphism $\phi: \Gamma \to \mathbf{R}$, where $\mathbf{R}$ is the additive group of reals, we say Γ is *ϕ-ordered* and put $P = \phi^{-1}([0, \infty))$. Also let $P_N = \phi^{-1}([N, \infty))$; γ is *ϕ-positive* (*ϕ-negative*) if $\phi(\gamma) > 0$ ($\phi(\gamma) < 0$). Again, $\mu \in M(G)$ is said to be of analytic type if $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus P$. We point out that the obvious analogue of Theorem 1 is valid with the same proof if the kernel of ϕ is finite. Repeating the proof of Corollary 1 we obtain the following result: If ker ϕ is finite and $\mu \in M(G)$ is of analytic type so is μ_s and $\hat{\mu}_s(\gamma) = 0$ for all $\gamma \in$ ker ϕ.

We say a subset $K \subset \Gamma$ is *ϕ-finite* if there exists a finite set $B \subset \mathbf{R}$ such that $\phi(K) \subset B$. A subset $K \subset \Gamma$ is *ϕ-bounded* if there exists a bounded set $B \subset \mathbf{R}$ such that $\phi(K) \subset B$.

We denote $(\ker \phi)^\perp$ by G_0. $M_{aG_0}(G)$ denotes the smallest L-ideal in $M(G)$ containing all measures of the form $m_{G_0} * \rho$ where m_{G_0} is Haar measure on G_0 and $\rho \in M(G)$. Put $M_{sG_0}(G) = M_{aG_0}(G)^\perp = \{\nu \in M(G)$: $\nu \perp \mu$ for each $\mu \in M_{aG_0}(G)\}$. For $\mu \in M(G)$ put $\mu = \mu_{aG_0} + \mu_{sG_0}$ where $\mu_{aG_0} \in M_{aG_0}(G)$ and $\mu_{sG_0} \in M_{sG_0}(G)$.

ϕ induces a continuous homomorphism ϕ^*: $\mathbf{R} \to G$ given by $\langle \phi^*(x), \gamma \rangle = e^{-i(x\phi(\gamma))}$ ($x \in \mathbf{R}$, $\gamma \in \Gamma$). Let Ψ: $M(\mathbf{R}) \to M(G)$ where $\Psi(\mu)(E) = \mu(\phi^{*-1}(E))$ and E is Borel in G. $M_{a\phi}(G)$ denotes the smallest L-ideal in $M(G)$ containing all measures of the form $\Psi(\omega) * \rho$ where $\omega \in M_a(\mathbf{R})$ and $\rho \in M(G)$. Put $M_{s\phi}(G) = M_{a\phi}(G)^{\perp} = \{\nu \in M(G)\colon \nu \perp \mu$ for each $\mu \in M_{a\phi}(G)\}$. For $\mu \in M(G)$ put $\mu = \mu_{a\phi} + \mu_{s\phi}$ where $\mu_{a\phi} \in M_{a\phi}(G)$ and $\mu_{s\phi} \in M_{s\phi}(G)$. It is easy to see that $M_{a\phi}(G)$ coincides with the measures $\mu \in M(G)$ that translate continuously in the direction of ϕ; i.e., $\lim_{x\to 0} \| \mu_{\phi^*(x)} - \mu \| = 0$.

K. de Leeuw and I. Glicksberg proved that if Γ is ϕ-ordered and $\mu \in M(G)$ is of analytic type then $\mu \in M_{a\phi}(G)$; see [3]. An easy consequence of the de Leeuw-Glicksberg result is: If $\mu \in M(G)$ is of analytic type and Γ is ϕ-ordered then μ_s is also of analytic type; see [3, p. 186]. We now show how to obtain these last two results with the methods of this paper.

PROPOSITION 2. *For $\mu \in M(G)$ and K ϕ-finite in Γ*

$$\lim_{K\to\infty} \| \mu \|_K = \| \mu_{sG_0} \|.$$

For $\mu \in M(G)$ and K ϕ-bounded in Γ

$$\lim_{K\to\infty} \| \mu \|_K = \| \mu_{s\phi} \|.$$

PROOF. The proof follows from easy modifications of [4]; see also [7].

THEOREM 2. *Let Γ be ϕ-ordered. Let $\nu \in M_{sG_0}(G)$ and $\lambda \in M_{aG_0}(G)$, or let $\nu \in M_{s\phi}(G)$ and $\lambda \in M_{a\phi}(G)$. Let ν be concentrated on Ω_ν and λ be concentrated on Ω_λ. Then given $\varepsilon > 0$ there is a trigonometric polynomial k having only ϕ-positive frequencies, $E \subset \Omega_\nu$, $F \subset \Omega_\lambda$ such that*

(a) $\|k\|_\infty < 1 + \varepsilon$;

(b) $|k - 1| < \varepsilon$ *on* $\Omega_\nu \setminus E$ *and* $|\nu|(E) < \varepsilon$;

(c) $|k| < \varepsilon$ *on* $\Omega_\lambda \setminus F$ *and* $|\lambda|(F) < \varepsilon$.

Furthermore, in the case $\nu \in M_{s\phi}(G)$ and $\lambda \in M_{a\phi}(G)$ we may choose k to have frequencies in P_N for any choice of N.

PROOF. The proof is to interpret the argument of Theorem 1 in the ϕ-ordered context.

First, suppose $\nu \in M_{sG_0}(G)$ and $\lambda \in M_{aG_0}(G)$. We may take λ to be nonnegative. $\lambda \ll m_{G_0} * \lambda$. Thus suppose $\hat{\lambda}$ vanishes off $\ker \phi$ and in addition that $\|\lambda\| = 1$. In the choice of the trigonometric polynomials $p_1, \ldots, p_Q$ replace (1.6) by

$$\bar{p}_s p_t \text{ has frequencies in } \Gamma \setminus \ker \phi \text{ for } s < t. \tag{2.1}$$

This can be done by Proposition 2. The remainder of the argument is repeated with m_G replaced by λ. The crucial observation is that

$$\int_G u_1(\theta)\, d\lambda(\theta) = 0$$

since k has no frequencies in $\ker \phi$ and $\operatorname{supp} \hat{\lambda} \subset \ker \phi$.

Now suppose $\nu \in M_{s\phi}(G)$ and $\lambda \in M_{a\phi}(G)$. Take λ to be nonnegative. Let $f \in M(\mathbf{R})$, $f \geqslant 0$, have compactly supported transform. Then $\lambda \ll \Psi(f) * \lambda$ and we

may thus suppose $\hat{\lambda}$ has ϕ-bounded support. Put $\|\lambda\| = 1$ and fix N in $\mathbf{Z}^+$ with $\operatorname{supp} \hat{\lambda} \subset \Gamma \setminus P_N$. In the choice of the trigonometric polynomials $p_1, \ldots, p_Q$ replace (1.6) by

$$\bar{p}_s p_t \text{ has frequencies in } P_N \cup (-P_N) \text{ for } s < t. \tag{2.2}$$

This can be done by Proposition 2. The remainder of the argument is repeated with m_G replaced by λ. The crucial observation is that

$$\int_G u_1(\theta)\, d\lambda(\theta) = 0$$

since k has its frequencies in P_N.

COROLLARY 3. *Let $\mu \in M(G)$ be of analytic type for Γ having a ϕ-ordering. Then*
(a) $\mu_{\phi s} = 0$,
(b) $\hat{\mu}_{G_0 s}(\gamma) = 0$ *for all* γ *with* $\phi(\gamma) \leqslant 0$,
(c) $\hat{\mu}_s(\gamma) = 0$ *for all* γ *with* $\phi(\gamma) < 0$.

PROOF. Put $\mu = \mu_{\phi a} + \mu_{\phi s}$ and let $\gamma_0 \in \Gamma$. Select from Theorem 2 a sequence $\langle k_n \rangle_1^\infty$ of trigonometric polynomials having frequencies in P_N where $|\phi(\gamma_0)| < N$ such that $k_n \to 1$ a.e. mod $|\mu_{\phi s}|$ and $k_n \to 0$ a.e. mod $|\mu_{\phi a}|$. Then as in the proof of Corollary 1

$$|\hat{\mu}_{\phi s}(\gamma_0)| \leqslant \int_G |1 - k_n|\, d|\mu_{\phi s}| + \left| \int_G k_n(x)\gamma_0(-x)\, d(\mu - \mu_{\phi a}) \right|.$$

But $k_n \bar{\gamma}_0$ has only ϕ-positive frequencies since Freq $k_n \subset P_N$. So

$$|\hat{\mu}_{\phi s}(\gamma_0)| \leqslant \int_G |1 - k_n|\, d|\mu_{\phi s}| + \int_G |k_n|\, d|\mu_{\phi a}|.$$

Both summands on the right converge to 0 as $n \to \infty$.

Now put $\mu = (= \mu_{\phi a}) = \mu_{G_0 a} + \mu_{G_0 s}$. From Theorem 2 select a sequence $\langle k_n \rangle_1^\infty$ of trigonometric polynomials having ϕ-positive frequencies such that $k_n \to 1$ a.e. mod $|\mu_{G_0 s}|$ and $k_n \to 0$ a.e. mod $|\mu_{G_0 a}|$ and $\|k_n\|_\infty \leqslant 2$. Let $\gamma_0 \in \{\gamma\colon \phi(\gamma) \leqslant 0\}$. Then $\gamma_0 k_n$ has only ϕ-positive frequencies and the argument of the preceding paragraph repeats to give

$$\hat{\mu}_{G_0 s} = 0 \quad \text{on } \{\gamma\colon \phi(\gamma) \leqslant 0\}.$$

We now have $\mu_{G_0 a}$ is analytic. Write $\mu_{G_0 a} = \mu_a + \mu_{G_0 a, s}$ where $\mu_{G_0 a, s}$ is the singular part. It remains to see $\mu_{G_0 a, s}$ is analytic. But $\mu_{G_0 a, s} \ll m_{G_0} * |\mu_{G_0 a, s}| = \nu$ is a positive singular measure, and therefore by Proposition 1 we may choose for any given finite set $F \subset \Gamma_0$ a sequence of trigonometric polynomials $\langle k_n \rangle_1^\infty$ such that Freq $k_n \subset \Gamma_0 \setminus F$, $k_n \to 1$ a.e. mod $|\mu_{G_0 a, s}|$ and $\|k_n\|_\infty \leqslant 2$. Choose a trigonometric polynomial s so that $\|s - \mu_a\| < \varepsilon$. Let γ_0 be ϕ-negative. Determine $\langle k_n \rangle_1^\infty$ by the choice $F = \gamma_0 \overline{(\operatorname{Freq} s)} \cap \Gamma_0$. Now

$$\begin{aligned} \hat{\mu}_{\phi a, s}(\gamma_0) &= \int \bar{\gamma}_0 (1 - k_n)\, d\mu_{\phi a, s} + \int \bar{\gamma}_0 k_n\, d\mu_{\phi a, s} \\ &= \int \bar{\gamma}_0 (1 - k_n)\, d\mu_{\phi a, s} - \int \bar{\gamma}_0\, k_n\, d\mu_a \qquad \text{(by the analyticity of } \mu_{\phi a}). \end{aligned}$$

So $|\hat{\mu}_{\phi a,s}(\gamma_0)| \leq \int |1 - k_n| d|\mu_{\phi a,s}| + |\int \bar{\gamma}_0 k_n \, d\mu_a|$. $\int |1 - k_n| \, d|\mu_{\phi a,s}| \to 0$ as $n \to \infty$. And $|\int \bar{\gamma}_0 k_n \, d\mu_a| \leq |\int \bar{\gamma}_0 k_n s \, d\theta| + 2\varepsilon = 2\varepsilon$ (since $\|\gamma_0 k_n\|_\infty \leq 2$).

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Department of Mathematics, Kansas State University, Manhattan, Kansas 66506

Department of Mathematics, Illinois State University, Normal, Illinois 61761