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AN OSCILLATION CONDITION FOR DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

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ABSTRACT. In separate papers, D. L. Lovelady has related oscillation of solutions of certain linear differential equations of odd order $\geqslant 3$ and even order $\geqslant 4$ to oscillation of an associated second order equation. This paper presents a unified proof of Lovelady's results for equations of arbitrary order $\geqslant 3$. The results are somewhat more detailed and the equations need not be linear.

D. L. Lovelady [2], [3] has proved the following theorem.

THEOREM 1. *Suppose p is positive and continuous on* $[0, \infty)$, *and*

$$\int^{\infty} p(s)s^{n-3}\,ds < \infty, \tag{1}$$

where $n > 2$. Suppose also that the equation

$$w'' + \left(\frac{1}{(n-3)!}\int_t^{\infty}(s-t)^{n-3}p(s)\,ds\right)w = 0 \qquad (t > 0) \tag{2}$$

is oscillatory. Then every solution of

$$y^{(n)} + p(t)y = 0 \qquad (t > 0)$$

is oscillatory if n is even, while if n is odd and y is nonoscillatory, then $y', \dots, y^{(n-1)}$ tend monotonically to zero and y approaches a finite limit as $t \to \infty$.

Lovelady gave separate proofs of the two halves of this theorem, considering even n in [3] and odd n in [2]. In this paper we adapt his methods to study oscillation of solutions of

$$y^{(n)} + f(t, y) = 0 \qquad (t > 0) \tag{3}$$

under the following assumptions.

ASSUMPTION A. *Let n be any integer $\geqslant 3$ and let f be continuous on* $(0, \infty) \times (-\infty, \infty)$, *with*

$$f(t, y)/y \geqslant p(t) \geqslant 0 \qquad (y \neq 0) \tag{4}$$

where p is nonnegative and continuous on $(0, \infty)$ *and satisfies* (1).

As usual, we define an *extendible* solution of (3) to be one which exists on some interval (T, ∞). Such a solution is said to be *oscillatory* if it has a zero on every interval (T_1, ∞). Because of (4), a nonoscillatory solution y of (3) must satisfy the

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inequality $y(t)y^{(n)}(t) < 0$ for sufficiently large t. The following elementary lemma of Kneser [1] describes the possible asymptotic behavior of such functions.

LEMMA 1. *Suppose* $y(t) > 0$ *and* $y^{(n)}(t) < 0$ *for large* t. *Then*
(a) *If* n *is even, there is an even integer* k *such that*

$$y^{(j)}(t) > 0 \quad \text{for } t > t_0, 0 < j < k+1, \tag{5}$$

$$\lim_{t\to\infty} y^{(k+1)}(t) = \alpha \quad (\text{finite} > 0), \tag{6}$$

and

$$\lim_{t\to\infty} y^{(j)}(t) = 0, \qquad k+2 < j < n-1. \tag{7}$$

(b) *If* n *is odd, then either there is an odd integer* k *for which* (5), (6), *and* (7) *hold, or*

$$\lim_{t\to\infty} y^{(j)}(t) = 0, \qquad 1 < j \leqslant n-1, \tag{8}$$

and

$$\lim_{t\to\infty} y(t) = \gamma \quad (\text{finite} > 0). \tag{9}$$

The limits in (6), (7), (8), *and* (9) *are all approached monotonically for large* t. *Moreover, these assertions remain valid if the stated inequalities on* $y, y', \ldots, y^{(n)}$ *and* α *and* γ *are all reversed.*

For convenience, we define

$$q(t) = \int_t^\infty (t-s)^{n-3} p(s)\, ds,$$

which exists, because of (1).

The proofs of the next two theorems are adaptations of Lovelady's proofs.

THEOREM 2. *Suppose Assumption* A *holds and* k *is an integer such that* $n-k$ *is even and* $0 < k < n-4$. *Suppose also that the equation*

$$w'' + \frac{q(t)}{(n-k-3)!k!} w = 0 \qquad (t > 0) \tag{10}$$

is oscillatory. Then no nonoscillatory solution of (3) *can satisfy either* (5), (6), *and* (7) *as stated, or* (5), (6), *and* (7) *with the inequalities on* $y^{(j)}$ *and* α *reversed.*

PROOF. We will show that if (3) has an eventually positive solution y which satisfies (5) and (7), then (10) is nonoscillatory. From (3) and (7),

$$y^{(k+2)}(t) = \frac{1}{(n-k-3)!} \int_t^\infty (t-s)^{n-k-3} f(s, y(s))\, ds.$$

Therefore,

$$-y^{(k+2)}(t) > \frac{1}{(n-k-3)!} \int_t^\infty (s-t)^{n-k-3} p(s) y(s)\, ds \qquad (t > t_0) \tag{11}$$

because of (4), (5), and the assumption that $n - k$ is even. If $k \geq 1$, then (5) implies that

$$y(s) \geq \frac{1}{(k-1)!}\int_{t_0}^{s}(s-\lambda)^{k-1}y^{(k)}(\lambda)\,d\lambda \qquad (t \geq t_0). \tag{12}$$

Substituting this into (11) yields

$$-y^{(k+2)}(t) \geq \frac{1}{(n-k-3)!\,(k-1)!} \cdot \int_t^{\infty}(s-t)^{n-k-3}p(s)\int_t^{s}(s-\lambda)^{k-1}y^{(k)}(\lambda)\,d\lambda\,ds \qquad (t \geq t_0). \tag{13}$$

Since $y^{(k+1)}(t) \geq 0$ if $t \geq t_0$ (see (5)), $y^{(k)}(\lambda) \geq y^{(k)}(t)$ if $\lambda \geq t \geq t_0$; hence, (13) implies that

$$-y^{(k+2)}(t) \geq \frac{q(t)}{(n-k-3)!k!}y^{(k)}(t) \qquad (t \geq t_0).$$

Although we assumed that $k \geq 1$ in deriving this inequality, it can be seen from (11) that it also holds for $k = 0$.

Now define $v(t) = y^{(k+1)}(t)/y^{(k)}(t)$; then

$$v'(t) + v^2(t) = \frac{y^{(k+2)}(t)}{y^{(k)}(t)} \leq \frac{-q(t)}{(n-k-3)!k!} \qquad (t \geq t_0). \tag{14}$$

By a theorem of Wintner [5] (see also [4, p. 63]), the existence of a continuously differentiable function v which satisfies the inequality in (14) implies that (10) is nonoscillatory. This completes the proof in the case where $y(t) > 0$ for large t. Because of (4), the proof for the case where $y(t) < 0$ for large t is similar.

THEOREM 3. *Suppose Assumption* A *holds and k is an integer such that $n - k$ is even and* $1 \leq k \leq n-2$. *Suppose also that the equation*

$$w'' + \frac{q(t)}{(n-k-2)!\,(k-1)!}w = 0 \qquad (t > 0) \tag{15}$$

is oscillatory. Then the conclusion of Theorem 2 *holds.*

PROOF. Again we suppose that y is an eventually positive solution of (3) which satisfies (5) and (7). From (3), (6), and (7),

$$y^{(k+1)}(t) \geq \frac{1}{(n-k-2)!}\int_t^{\infty}(t-s)^{n-k-2}f(s, y(s))\,ds \qquad (t \geq t_0).$$

Because of (4) and the assumption that $n - k$ is even, this implies that

$$y^{(k+1)}(t) \geq \frac{1}{(n-k-2)!}\int_t^{\infty}(s-t)^{n-k-2}p(s)y(s)\,ds \qquad (t \geq t_0).$$

From this and (12),

$$y^{(k+1)}(t) \geq \frac{1}{(n-k-2)!\,(k-1)!} \cdot \int_t^{\infty}(s-t)^{n-k-2}p(s)\int_t^{s}(s-\lambda)^{k-1}y^{(k)}(\lambda)\,d\lambda\,ds \qquad (t \geq t_0).$$

Now define a sequence $\{u_m\}$ of functions on $[t_0, \infty)$ by $u_0(t) = y^{(k)}(t)$ and, for $m \geqslant 1$,

$$u'_m(t) = \frac{1}{(n-k-2)!\,(k-1)!} \cdot \int_t^\infty (s-t)^{n-k-2} p(s) \int_t^s (s-\lambda)^{k-1} u_{m-1}(\lambda)\, d\lambda\, ds \qquad (t > t_0),$$

$$u_m(t_0) = y^{(k)}(t_0). \tag{16}$$

By induction,

$$y^{(k)}(t_0) \leqslant u_{m+1}(t) \leqslant u_m(t) \leqslant y^{(k)}(t) \qquad (t \geqslant t_0, m \geqslant 0) \tag{17}$$

and

$$0 \leqslant u'_{m+1}(t) \leqslant u'_m(t) \leqslant y^{(k+1)}(t) \qquad (t \geqslant t_0, m \geqslant 0). \tag{18}$$

Therefore, $\{u_m\}$ is uniformly bounded and equicontinuous on finite subintervals of $[t_0, \infty)$, so the Arzela-Ascoli theorem and the monotonicity with respect to m imply that $u(t) = \lim_{m\to\infty} u_m(t)$ exists on $[t_0, \infty)$, that the convergence is uniform on finite subintervals, and that u is continuous. From (18), $\{u'_m\}$ also converges on $[t_0, \infty)$; moreover, it can be shown that the integrals in (16) converge uniformly on finite subintervals of $[t_0, \infty)$. Therefore, u is differentiable and

$$u'(t) = \frac{1}{(n-k-2)!\,(k-1)!} \cdot \int_t^\infty (s-t)^{n-k-2} p(s) \int_t^s (s-\lambda)^{k-1} u(\lambda)\, d\lambda\, ds \qquad (t > t_0). \tag{19}$$

If $k = n - 2$, differentiating this yields

$$u''(t) + \frac{q(t)}{(n-3)!} u(t) = 0 \qquad (t > t_0).$$

Since

$$u(t) \geqslant y^{(k)}(t_0) > 0 \qquad (t \geqslant t_0)$$

from (17), we have now shown that (15) (with $k = n - 2$) has a nonoscillatory solution, which contradicts our assumption and completes the proof for this case. If $1 \leqslant k < n - 2$, differentiating (19) yields

$$u''(t) + \frac{q(t)}{(n-k-2)!\,(k-1)!} u(t) = P(t) \qquad (t > t_0) \tag{20}$$

with

$$P(t) = \frac{-1}{(n-k-3)!\,(k-1)!} \int_t^\infty (s-t)^{n-k-3} p(s) \int_t^s (s-\lambda)^{k-1} u(\lambda)\, d\lambda\, ds < 0. \tag{21}$$

Thus, $u(t) > 0$ on $[t_0, \infty)$ and the left side of (20) is negative. We will show that this contradicts our assumption that (15) is oscillatory.

Let t_1 and t_2 be successive zeros of a nontrivial solution w of (15), with $t_0 < t_1 < t_2$, and assume without loss of generality that

$$w(t) > 0 \qquad (t_1 < t < t_2).$$

Then

$$w'(t_1) > 0, \quad w'(t_2) < 0. \tag{22}$$

If $W = uw' - u'w$, then (22) implies that

$$W(t_1) = u(t_1)w'(t_1) > 0 \quad \text{and} \quad W(t_2) = u(t_2)w'(t_2) < 0. \tag{23}$$

However, from (15), (20), and (21),

$$W'(t) = -P(t)w(t) > 0 \quad (t_1 < t < t_2),$$

which contradicts (23). This completes the proof.

If $1 \leqslant k \leqslant n - 4$, then Sturm's comparison theorem implies that Theorem 3 is stronger than Theorem 2 when $2k < n - 2$, and that the opposite is true when $2k > n - 2$.

The following theorem is our main result.

THEOREM 4. *Suppose Assumption* A *holds and* (2) *is oscillatory. Then* (a) *if n is even, every extendible solution of* (3) *is oscillatory*; (b) *if n is odd and y is an extendible nonoscillatory solution of* (3), *then $y', \ldots, y^{(n-1)}$ tend monotonically to zero as $t \to \infty$, and* $\lim_{t\to\infty} y(t) = \gamma$ (*finite*). *In this case,* $\gamma = 0$ *if*

$$\int^{\infty} s^{n-1}p(s)\, ds = \infty. \tag{24}$$

PROOF. Since $m!n! < (m + n)!$ for any positive integers m and n, Sturm's comparison theorem and the assumption that (2) is oscillatory imply that (10) and (15) are oscillatory for every admissible k. Therefore, Lemma 1 and Theorems 2 and 3 imply all the conclusions of this theorem except that $\gamma = 0$ if (24) holds. To see the latter, note that if y is a solution of (3) which satisfies (8) and (9), then

$$y(t) = \gamma + \frac{1}{(n-1)!}\int_t^{\infty} (s - t)^{n-1} f(s, y(s))\, ds. \tag{25}$$

If $\gamma > 0$, then $f(s, y(s)) \geqslant \gamma p(s)$ for large s, so the existence of the integral in (25) contradicts (24). The conclusion follows in a similar way if $\gamma < 0$. This completes the proof.

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