II-REGULAR VARIATION

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Abstract. A function \( U: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is said to be \( \Pi \)-regularly varying with exponent \( \alpha \) if \( U(x)x^{-\alpha} \) is nondecreasing and there exists a positive function \( L \) such that
\[
\frac{U(\lambda x)/\lambda^\alpha - U(x)}{x^\alpha L(x)} \rightarrow \log \lambda \quad (x \rightarrow \infty) \text{ for } \lambda > 0.
\]
Suppose
\[
\hat{U}(t) \doteq \int_0^\infty e^{-ts} \, dU(x) \text{ exists for } t > 0.
\]
We prove that \( U \) is \( \Pi \)-regularly varying iff \( \hat{U} \) is \( \Pi \)-regularly varying.

1. Introduction. First we give the definition of regular variation.

Definition. A function \( U \) is said to be regularly varying with exponent \( \rho \) at infinity if it is real-valued, positive and measured on \((0, \infty)\) and if for each \( \lambda > 0 \)
\[
\lim_{x \rightarrow \infty} \frac{U(\lambda x)}{U(x)} = \lambda^\rho \quad \text{where } \rho \in \mathbb{R} \text{ (notation } U(x) \in RV_\rho). \]

Regularly varying functions with exponent zero are called slowly varying. The theory of regularly varying functions has been developed by Karamata. For some basic facts see [1], [8], [9].

A recent treatment of regular variation is also given in Seneta’s book [10]. Karamata proved the following theorems on regular variation which are basic in this theory.

Theorem A. Suppose \( U: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is Lebesgue summable on finite intervals.
(i) If \( U \) varies regularly at infinity with exponent \( \beta > -1 \) then
\[
\lim_{x \rightarrow \infty} \frac{xU(x)}{\int_0^x U(t) \, dt} = \beta + 1.
\]
(ii) If \( \lim_{x \rightarrow \infty} (xU(x)/\int_0^x U(t) \, dt) = \beta + 1 \) with \( \beta > -1 \) then \( U(x) \in RV_\beta \).

See, e.g., [5, Theorem 1.2.1].

The second theorem concerns the Laplace-Stieltjes transform: \( \hat{U}(t) = \int_0^\infty e^{-ts} \, dU(s) \) of \( U \). For a proof of this theorem the reader is referred to [10, Theorem 2.3].

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Theorem B. Suppose \( U: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is nondecreasing, right-continuous \( U(0^+) = 0 \), \( \hat{U}(t) \) is finite for \( t > 0 \). For \( \beta > 0 \) the following assertions are equivalent:

(i) \( U(x) \in RV_\beta \);
(ii) \( \hat{U}(1/x) \in RV_\beta \).

Both imply

(iii) \( \lim_{x \to \infty} \left( \frac{U(x)}{\hat{U}(1/x)} \right) = 1/\Gamma(\beta + 1) \).

For nondecreasing functions \( U \) we can combine Theorems A and B using the notion of a fractional integral:

Definition. \( \alpha U(x) = \left( \frac{1}{\Gamma(\alpha + 1)} \right) \int_0^x (x - t)^\alpha \, dU(t) \) where \( \alpha > 0 \).

Theorem C. Suppose \( U: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is nondecreasing and right-continuous, \( U(0^+) = 0 \) and \( \hat{U}(t) \) is finite for \( t > 0 \). For \( \alpha > 0 \) and \( \beta > 0 \) the following assertions are equivalent:

(i) \( U(x) \in RV_\beta \);
(ii) \( \alpha U(x) \in RV_{\alpha + \beta} \);
(iii) \( \hat{U}(1/x) \in RV_\beta \).

They imply

(iv) \( \alpha U(x)/x^\alpha U(x) \to \Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1) \) \( (x \to \infty) \);
(v) \( U(x)/\hat{U}(1/x) \to 1/\Gamma(\beta + 1) \) \( (x \to \infty) \).

Remark that the case \( \alpha = 1 \) yields Theorem A(i) with \( \beta > 0 \). For arbitrary \( \alpha > 0 \) Theorem C can be proved by using Theorems A and B and the relation

\[ \alpha \hat{U}(1/x) = x^\alpha \hat{U}(1/x), \]

since \( \alpha U(x) \) is nondecreasing.

In 1963 Bojanic and Karamata [2] studied the class of functions \( U \) for which

\[ \lim_{x \to \infty} \frac{U(\lambda x) - U(x)}{x^\sigma L(x)} \]

exists for some function \( L(x) \) and showed that \( \sigma \) can be chosen such that \( L(x) \) is slowly varying. In this paper we shall see that the Theorems A and B can be sharpened for functions \( U \) which satisfy the relation

\[ \lim_{x \to \infty} \frac{U(\lambda x)/\lambda^\sigma - U(x)}{x^\sigma L(x)} = \log \lambda \]

for some function \( L(x) \) and \( \sigma > 0 \) fixed. For \( \sigma = 0 \) this relation defines the class \( \Pi \).

Theorem D. Suppose \( \phi: \mathbb{R}^+ \to \mathbb{R} \) is nondecreasing. Then the following three statements are equivalent:

(i) There exist functions \( a: \mathbb{R}^+ \to \mathbb{R}^+ \) and \( b: \mathbb{R}^+ \to \mathbb{R} \) such that for all positive \( x \)

\[ \lim_{t \to \infty} \frac{\phi(tx) - b(t)}{a(t)} = \log x; \]

(ii) there exists a slowly varying function \( L \) such that

\[ \phi(x) = L(x) + \int_1^x L(t)/t \, dt; \]
(iii) there exists a slowly varying function $L_0$ such that
\[ \phi(x) = L_0(x) + \int_0^x L_0(t)/t \, dt. \]

Moreover if a function $\phi$ satisfies the conditions of this theorem then
\[ a(x) \sim L(x) \sim \phi(x) - \phi(x) \sim \frac{1}{x} \int_0^x s \, d\phi(s) \sim L_0(x) \quad (x \to \infty) \]
(see [5, Theorem 1.4.1]).

We call the function $a(x)$ the auxiliary function of $\phi(x)$. This function is (of course) determined up to asymptotic equivalence.

**Definition.** A function $\phi$ which satisfies the conditions of Theorem D is said to belong to the class $\Pi$. It can be shown that the class $\Pi$ is a proper subclass of the slowly varying functions (see [5, Corollary 1.4.1]). From Theorem D we can see that if $\phi(x) \in \Pi$ with auxiliary functions $a(x)$ and $[\phi(x) - \phi_1(x)]/a(x) \to c \quad (x \to \infty)$ where $c \in R$ is a constant and $\phi_1(x)$ a nondecreasing function, then $\phi_1(x) \in \Pi$ with auxiliary function $a(x).

In this paper we generalize the following theorem (see [6]).

**Theorem E.** Suppose $\phi: R^+ \to R^+$ is nondecreasing, $\phi(0 + ) = 0$ and $\hat{\phi}(s)$ is finite for $s > 0$. Then the following statements are equivalent:

(i) $\phi(x) \in \Pi$;
(ii) $\hat{\phi}(1/x) \in \Pi$;

Both imply
(iii) $\left(\phi(x) - \hat{\phi}(1/x)\right)/(1/x)\int_0^x s \, d\phi(s) \to \gamma \quad (x \to \infty)$.

We give a second order version of Karamata's Theorems A and B for nondecreasing functions $U$. A necessary and sufficient condition for a function to obey the second order relation is formulated in the following definition.

**Definition.** $U \in \Pi RV_\alpha$ iff $U(x)/x^\alpha \in \Pi$ where $\alpha \in R$.

If $U \in \Pi RV_\alpha$ then we say that $L$ is the auxiliary function of $U$ if $L$ is the auxiliary function of $U(x)/x^\alpha \in \Pi$. We call the function $U$ $\Pi$-regularly varying with exponent $\alpha$. The $\Pi$-varying functions with exponent $\alpha$ form a subclass of $RV_\alpha$.

2. Results. Our result is the following theorem.

**Theorem 1.** Suppose $\alpha > 0$, $\beta > 0$, $U: R^+ \to R^+$, $U(x)/x^\beta$ nondecreasing, $\lim_{x \to 0}(U(x)/x^\beta) = 0$, and $\hat{U}(t)$ exists for $t > 0$. Then the following statements are equivalent:

(i) $U(x) \in \Pi RV_\beta$;
(ii) $U(x) \in \Pi RV_{\alpha + \beta}$;
(iii) $U(1/x) \in \Pi RV_\beta$.

They imply
\[ \frac{(\Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1))U(x) - aU(x)/x^\alpha}{x^{\beta - 1}\int_0^x s \, d(U(s)/s^\beta)} \to - \frac{\hat{\alpha}}{\hat{\beta}} \left( \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} \right) \quad (x \to \infty), \]
\((v)\) \[ \frac{U(x) - (1/\Gamma(\beta + 1)) \tilde{U}(1/x)}{x^{\beta - 1} \int_0^x s \, d(U(s)/s^\beta)} \to -\psi(\beta + 1) \quad (x \to \infty) \]

where \(\psi(x) = (d/dx)\log \Gamma(x)\).

Conversely if (iv) with \(\alpha \in (0, 1], \beta > 0\) then (i) and if (v) with \(\beta > 1\) then (i).

**Proof.** (i) \(\to\) (iv) and (i) \(\to\) (ii). We write

\[ U(x) = x^\beta \left( L(x) + \int_0^x \frac{L(t)}{t} \, dt \right) \]

with

\[ L(x) = \frac{1}{x} \int_0^x s \, d\frac{U(s)}{s^\beta} \in RV_\delta^{(\infty)}. \]

Then

\[ \left( \alpha U(x)/x^\alpha \right) - (\Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1)) U(x) \]

\[ \frac{x^\beta L(x)}{x^{\beta - 1} \int_0^x s \, d(U(s)/s^\beta)} \]

\[ = \frac{1}{\Gamma(\alpha)} \int_0^1 (1 - t)^{\alpha - 1} t^{\beta} \left( \frac{U(tx)/t^\beta x^\beta}{L(x)} - \left( \frac{U(x)}{x^\beta} \right) \right) \, dt \]

\[ \to \frac{1}{\Gamma(\alpha)} \int_0^1 \log(1 - t)^{\alpha - 1} t^{\beta} \, dt \quad (x \to \infty). \]

The last step is justified since by substituting the expression for \(U(x)\) we find

\[ \frac{1}{\Gamma(\alpha)} \left[ \int_0^1 (1 - t)^{\alpha - 1} t^{\beta} \left\{ \frac{L(tx)}{L(x)} - 1 \right\} \, dt - \int_0^1 (1 - t)^{\alpha - 1} t^{\beta} \int_t^1 \frac{L(sx)}{L(x)} \, ds \, dt \right] \]

and

\[ \frac{s^x x^\delta L(sx)}{x^\gamma L(x)} \to s^\delta \quad (x \to \infty) \]

uniformly on \((0, 1)\) where \(\varepsilon > 0\) (see de Haan [5]). Now (iv) and (i) imply (ii) as mentioned in the introduction.

(i) \(\to\) (v) and (i) \(\to\) (iii). We write \(U(x) = x^\beta L(x) + K(x)\) where \(K(x) = x^\beta \int_0^L(L(t)/t) \, dt\). By Karamata’s Theorem B we have

\[ x^\beta L(x) - \frac{1}{\Gamma(\beta + 1)} \int_0^\infty e^{-t/x} \, d(t^\beta L(t)) = o(x^\beta L(x)) \quad (x \to \infty). \]

Substituting the expression for \(K(x)\) we find

\[ \frac{K(x) - \hat{K}(1/x)/\Gamma(\beta + 1)}{x^\beta L(x)} = \int_0^1 \frac{L(tx)}{L(x)} \, dt \]

\[ - \frac{1}{\Gamma(\beta + 1)} \int_0^\infty e^{-t^\beta} \int_0^t \frac{L(ux)}{L(x)} \, du \, dt \]

\[ = - \frac{1}{\Gamma(\beta + 1)} \int_0^1 e^{-t^\beta} \int_1^t \frac{L(ux)}{x^\beta L(x)} \, du \, dt \]

\[ \quad - \frac{1}{\Gamma(\beta + 1)} \int_1^\infty e^{-t^\beta} \int_1^t \frac{L(ux)}{x^\beta L(x)} \, du \, dt = (*) \]
since \( \Gamma(\beta + 1) = \int_0^\infty e^{-t^\beta} \, dt \). Since
\[
u^x L(ux) \xrightarrow{x \to \infty} u^x \quad \text{uniformly on } (0, 1)
\]
and
\[
u^{-x} L(ux) \xrightarrow{x \to \infty} u^{-x} \quad \text{uniformly on } (1, \infty)
\]
(see [5, Corollary 1.2.1.4]) we find
\[
(*) \rightarrow -\frac{1}{\Gamma(\beta + 1)} \int_0^\infty e^{-t^\beta} \log t \, dt = -\psi(\beta + 1) \quad (x \to \infty).
\]
This proves (i) \(\rightarrow\) (v). Now we have analogously that (v) and (i) imply (iii).

(ii) \(\rightarrow\) (iii) follows immediately since \(\tilde{U}(1/x) = x^\beta \tilde{U}(1/x)\) and we can use (i) \(\rightarrow\) (iii).

(i) \(\leftrightarrow\) (iii). Writing \(V(x) = U(x)/x^\beta\) we have by Proposition P4 in [7]
\[
V(x) \in \Pi \text{ iff } \int_0^x t^\beta \, dV(t) \in RV_\beta \quad \text{where } \beta > 0.
\]
Or
\[
U(x) \in \Pi RV_\beta \text{ iff } U(x) - \int_0^x \frac{U(t)}{t} \, dt \in RV_\beta.
\]
This is equivalent to
\[
\tilde{U}(1/x) - \beta x \tilde{K}(1/x) \in RV_\beta \quad \text{where } K(x) = U(x)/x.
\]
The last statement is equivalent to \(\tilde{U}(1/x) \in \Pi RV_\beta\), since \(x \tilde{K}(1/x) = \int_0^x (\tilde{U}(1/t)/t) \, dt\). (Both sides have the same derivative.) The case \(\beta = 0\) is the result of Theorem E.

(iv) \(\rightarrow\) (i). As in the proof of (i) \(\rightarrow\) (iv) we write
\[
U(x) = x^\beta \left\{ L(x) + \int_0^x \frac{L(t)}{t} \, dt \right\}.
\]
Substituting this expression in (iv) and rearranging we see that (iv) is equivalent to
\[
\frac{1}{\Gamma(\alpha)} \int_0^x \left(1 - \frac{t}{x}\right)^{\alpha - 1} \frac{1}{x} L(t) \, dt + \frac{1}{\Gamma(\alpha)} \int_0^x \int_t^x (1 - u)^{\alpha - 1} u^\beta \, dL(u) \, \frac{dt}{t}
\]
\[
- \frac{1}{\Gamma(\alpha + \beta + 1)} \int_0^x L(t) \, \frac{dt}{t} \sim \xi L(x) \quad (x \to \infty),
\]
where \(\xi = (\Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1)) + (d/d\beta)(\Gamma(\beta + 1)/\Gamma(\alpha + \beta + 1)).\) Or
\[
\int_0^\infty L(t) k(x/t) \, dt/t \sim \xi L(x) \quad (x \to \infty)\] where the kernel \(k\) is defined by
\[
k(1/x) = \frac{x}{\Gamma(\alpha)} \left\{ (1 - x)^{\alpha - 1} x^\beta - \frac{1}{x} \int_0^x (1 - u)^{\alpha - 1} u^\beta \, du \right\}
\]
for \(x < 1\) and 0 for \(x > 1\). For \(\alpha \in (0, 1]\) and \(\beta > 0\) the kernel is nonnegative since \((1 - x)^{\alpha - 1} x^\beta\) is increasing on \((0, 1)\). Moreover we have \(\lim_{x \to \infty; t \to 1^+} \inf(L(tx)/L(x)) > 1\) since \(xL(x)\) is nondecreasing. Application of Theorem 6.2 in [3] then gives the result since \(\tilde{k}(\rho) = \int_0^1 k(1/t) t^{\rho - 1} \, dt\) is decreasing for \(\rho > -\beta - 1\) and so \(\tilde{k}(\rho) = \xi\) only if \(\rho = 0\).
(v) \implies (i). We define $L(x)$ as in the proof of (iv) \implies (i). Here we can reformulate (v) as follows:

$$\int_0^\infty k\left(\frac{x}{t}\right)L(t)\frac{dt}{t} \sim \xi L(x) \quad (x \to \infty),$$

where $\xi = 1 + \psi(\beta + 1)$ and the kernel $k$ is given by

$$k\left(\frac{1}{x}\right) = \frac{1}{\Gamma(\beta + 1)} x^\beta e^{-x} - \frac{1}{x} \int_0^x u^\beta e^{-u} du + 1 - I_{(0,1)}(x).$$

If $\beta > 1$ this kernel is positive for all $x > 0$, since the term $x^\beta e^{-x}$ is increasing on $(0, \beta)$. Here we can also apply Theorem 6.2 in [3].

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REFERENCES

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