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A TAUBERIAN PROBLEM FOR A VOLTERRA INTEGRAL OPERATOR

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ABSTRACT. The following question is studied: For which (nonintegrable) kernels A does $\lim_{t\to\infty} \int_0^t A(t-s)x(s)\,ds = 0$ imply that $\lim_{t\to\infty} x(t) = 0$ when x is bounded and satisfies a Tauberian condition.

1. Introduction. The purpose of this paper is to study under what conditions on the kernel A it follows from $\lim_{t\to\infty} \int_0^t A(t-s)x(s)\,ds = 0$ that $\lim_{t\to\infty} x(t) = 0$ when x is bounded and satisfies a Tauberian condition. If $A \in L^1(\mathbf{R}^+; \mathbf{C})$, $\mathbf{R}^+ = [0, \infty)$, then the answer is immediately given by Pitt's form of Wiener's Tauberian theorem, see [**10**, p. 210], that is, the Fourier transform of A should not vanish anywhere. For this reason we will really only consider nonintegrable kernels A, but we will, of course, always assume that $A \in L^1_{\mathrm{loc}}(\mathbf{R}^+; \mathbf{C})$ so that the integral $\int_0^t A(t-s)x(s)\,ds$ is well defined. This would not be possible without the restriction to functions supported on $\mathbf{R}^+$.

In Theorem 1 it is shown how one can get around the problem with the nonintegrability of A by introducing a "resolvent" kernel. But this result only gives sufficient conditions, which are not too easy to verify. In Theorem 2 however, both necessary and sufficient conditions are given for certain special real kernels (sums of an integrable function and a nonnegative and nonincreasing one). The condition used involves the behavior of the Fourier transform of A and does not explicitly involve "resolvent" kernels, although such a formulation would be possible.

2. Statement of results. We write $(f * g)(t) = \int_0^t f(t-s)g(s)\,ds$ and let "$\hat{}$" denote the Fourier transform, i.e. $A\hat{}(z) = \int_0^\infty e^{-izt}A(t)\,dt$ (with the integral extended over $(-\infty, \infty)$ when the function is defined on $\mathbf{R}$). The proof of the following theorem is due to O. J. Staffans.

THEOREM 1. *Assume that*

$$e^{-\sigma t}A(t) \in L^1(\mathbf{R}^+; \mathbf{C}), \qquad \sigma > 0, \tag{2.1}$$

$$\limsup_{v\to 0-} |A\hat{}(u+iv)| > 0 \quad \textit{for all } u \in \mathbf{R}; \tag{2.2}$$

there exists a complex number c, $c \neq 0$, such that

$$r_c \in L^1(\mathbf{R}^+; \mathbf{C}) \tag{2.3}$$

where

$$cr_c(t) + (A * r_c)(t) = A(t), \qquad t \in \mathbf{R}^+. \tag{2.4}$$

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Now if

$$x \in L^\infty(\mathbf{R}^+; \mathbf{C}) \tag{2.5}$$

is such that

$$\lim_{t\to\infty, \tau\to 0} |x(t+\tau) - x(t)| = 0 \tag{2.6}$$

and

$$\lim_{t\to\infty} (A * x)(t) = 0, \tag{2.7}$$

then

$$\lim_{t\to\infty} x(t) = 0. \tag{2.8}$$

To prove this theorem one observes that

$$(r_c * x)(t) = c^{-1}((A * x)(t) - (r_c * A * x)(t)) \to 0$$

as $t \to \infty$ by (2.3), (2.4) and (2.7) since the convolution of an integrable function and a bounded function converging to 0, converges to 0. But as $\hat{r_c}(z) = \hat{A}(z)(c + \hat{A}(z))^{-1}$ one sees from (2.2) that $\hat{r_c}(u) \neq 0$, $u \in \mathbf{R}$, and one can apply Pitt's form of Wiener's Tauberian theorem.

In general it is very difficult to see if (2.3) is satisfied, but note that if $A \in L^1(\mathbf{R}^+; \mathbf{C})$, then (2.3) holds if $|c| > \|A\|_{L^1(\mathbf{R}^+;\mathbf{C})}$. Some conditions on A that do imply (2.3) can be found in [3], [4] and [8]. With the aid of results in these references one can conclude that the crucial condition (2.3) holds at least in the following two cases (more complicated examples could be given).

PROPOSITION. *Assume that*

$$A(t) = A_1(t) + A_2(t),\ t \in \mathbf{R}^+, \quad \textit{where } A_1\colon \mathbf{R}^+ \to \mathbf{R}^+ \textit{ is non-increasing and convex and } A_2 \in L^1(\mathbf{R}^+; \mathbf{C}), \tag{2.9}$$

or

$$A(t) = A_1(t) + A_2(t),\quad t \in \mathbf{R}^+, \quad \textit{where } A_1 \in BV(\mathbf{R}^+; \mathbf{C}),\ \lim_{t\to\infty} A_1(t) = A_1(\infty) \neq 0 \textit{ and } A_2 \in L^1(\mathbf{R}^+; \mathbf{C}). \tag{2.10}$$

Then the function A satisfies (2.3).

If (2.9) holds, then c and if (2.10) holds, then $c/A_1(\infty)$ are chosen to be sufficiently large positive numbers so that $c + \hat{A}(z) \neq 0$, $\operatorname{Im} z < 0$ (see [8, p. 320]).

It is reasonable to ask to what extent (2.3) is necessary for the conclusion of Theorem 1. In the following theorem this problem is considered for a certain class of kernels but there (2.3) is replaced by a condition on the Fourier transform of A.

THEOREM 2. *Assume that*

$$A(t) = A_1(t) + A_2(t), \qquad t \in \mathbf{R}^+, \tag{2.11}$$

where

$$A_1\colon \mathbf{R}^+ \to \mathbf{R}^+ \textit{ is nonincreasing} \tag{2.12}$$

and

$$A_2 \in L^1(\mathbf{R}^+; \mathbf{C}), \qquad \int_0^\infty (\operatorname{Im} A_2(t))\, dt = 0. \tag{2.13}$$

Then (2.5)–(2.7) *imply* (2.8) *if and only if*

$$\text{for every } u_0 \in \mathbf{R} \text{ there exists a function } k \in L^1(\mathbf{R}; \mathbf{C}) \text{ and a number } \delta > 0 \text{ so that } k\hat{}(u) = \lim_{v\to 0-}(A\hat{}(u + iv))^{-1},\ |u - u_0| < \delta. \tag{2.14}$$

Observe that if A is integrable, then (2.14) is equivalent to $A\hat{}(u) \neq 0$, $u \in \mathbf{R}$, see [**10**, p. 207].

Since A_1 is of bounded variation one sees that $A\hat{}(z)$ is continuous in $\operatorname{Im} z \leqslant 0$ everywhere except perhaps at $z = 0$ and moreover one can easily deduce that (2.14) is equivalent to the assumption that (2.2) holds and

$$\text{there exists a function } k_1 \in L^1(\mathbf{R}; \mathbf{C}) \text{ and a number } \delta > 0 \text{ so that } k_1\hat{}(u) = \lim_{v\to 0-}(A_1\hat{}(u + iv))^{-1},\ |u| < \delta. \tag{2.15}$$

This follows from the fact that since A_1 is of bounded variation, its Fourier transform $A_1\hat{}(u)$ is everywhere, except perhaps at 0, locally equal to the Fourier transform of an integrable function, cf. [**10**, pp. 202–210]. Note that (2.14) is a consequence of (2.2) and (2.3) and one can prove (using the argument in [**6**, pp. 60–63]), that (2.3) holds if (2.1) and (2.14) hold, $\lim_{v\to 0-} A\hat{}(u + iv)$ is for sufficiently large $|u|$ equal to the Fourier transform of a function in $L^1(\mathbf{R}; \mathbf{C})$ with small norm and there exists a number $c \neq 0$ such that $A\hat{}(z)(c + A\hat{}(z))^{-1}$ is bounded and continuous in $\operatorname{Im} z \leqslant 0$.

Finally we remark that a slight modification of the example in [**2**], e.g. add e^{-t} to the kernel there, gives a nonnegative, nonincreasing function A such that (2.2) holds but (2.14) does not hold, i.e. (2.5)–(2.7) do not imply (2.8) in this case.

3. Proof of Theorem 2. First we consider the relatively simple case when $\lim_{t\to\infty} A_1(t) = A_1(\infty) > 0$. If (2.2) holds, then it follows from Theorem 1 and the Proposition that (2.5)–(2.7) imply (2.8) and moreover that $r_c \in L^1(\mathbf{R}^+; \mathbf{C})$ and $r_c\hat{}(u) \neq 0$, $u \in \mathbf{R}$. Now we get (2.14) because $(A\hat{}(z))^{-1} = c^{-1}((r_c\hat{}(z))^{-1} - 1)$, see [**10**, p. 207]. If (2.2) does not hold then $A\hat{}(u_0) = 0$ for some real $u_0 \neq 0$ and clearly (2.14) fails. Let $x(t) = e^{iu_0 t}\min\{1, t|u_0|/(2\pi)\}$, $t \in \mathbf{R}^+$. This function is of course uniformly continuous and bounded but does not satisfy (2.8). Hence we must check that $(A * x)(t) \to 0$ as $t \to \infty$ and this is true because $e^{-iu_0 t}(A * x)(t) \to A\hat{}(u_0) = 0$, as is seen when one performs an integration by parts in the term with A_1. This completes the proof in the case when $A_1(\infty) > 0$.

Next we assume that $A_1(\infty) = 0$ and that $A_1 \notin L^1(\mathbf{R}^+; \mathbf{R})$, because otherwise the result we want to prove is just Pitt's form of Wiener's Tauberian theorem. Since we can modify the representation of A as the sum of A_1 and A_2 and multiply A by any nonzero number, we may in addition to (2.12) and (2.13) assume that

$$A_1 \text{ is continuously differentiable on } \mathbf{R}^+, \quad A_1(0) = 1, \tag{3.1}$$

and

$$\int_0^\infty A_2(s)\,ds = 0. \tag{3.2}$$

To see this proceed as follows: First modify A_1 so that $A_1 \in C^1(\mathbf{R}^+;\ \mathbf{R}^+)$ *and* $A_1'(T) = 0$ where $\int_0^T tA_1'(t)\,dt < -|\int_0^\infty \operatorname{Re} A_2(t)\,dt|$. Then add to A_1 and subtract from A_2 the function

$$\left(\int_0^\infty \operatorname{Re} A_2(t)\,dt \Big/ \left(-\int_0^T tA_1'(t)\,dt\right)\right)\max\{0, A_1(t) - A_1(T)\}.$$

We observe that $A\hat{}(u) \neq 0$, $u \in \mathbf{R}$, is a necessary condition for (2.14) and for (2.5)–(2.7) to imply (2.8) since the proof for the case $A_1(\infty) > 0$ still works. Thus we may from now on assume that $A\hat{}(u) \neq 0$, $u \in \mathbf{R}$, $(A\hat{}(0) = +\infty)$. As already noted in §2 it is with this assumption possible to conclude that (2.14) is equivalent to (2.15) because the only problems occur at the origin and there $A_1\hat{}(u) + A_2\hat{}(u) = A_1\hat{}(u)(1 + A_2\hat{}(u)/A_1\hat{}(u))$ and $1/A_1\hat{}(0) = 0$, see [**10**, pp. 202–210].

The next step is to show that one can replace (2.7) by

$$\lim_{t\to\infty} (A_1 * x)(t) = 0. \tag{3.3}$$

Let

$$q_n(t) = \max\{0, \min\{tn^2, 2n - tn^2\}\}, \qquad t \in \mathbf{R}^+, n = 1, 2, \ldots.$$

If $x \in L^\infty(\mathbf{R}^+;\ \mathbf{C})$, then we define the functions y_n, $n = 1, 2, \ldots,$ by

$$y_n(t) = (q_n * x)(t), \qquad t \in \mathbf{R}^+. \tag{3.4}$$

It is easy to check that if (2.5) and (2.6) hold and $\lim_{t\to\infty} y_n(t) = 0$ for all n, then (2.8) holds. Suppose that (2.5)–(2.7) hold and fix $n \geq 1$. From (2.7) and (3.4) we see that $\lim_{t\to\infty}(A * y_n)(t) = 0$ and since the function $A * y_n$ is differentiable with uniformly continuous derivative, we deduce that the derivative must converge to 0 too. That is

$$\lim_{t\to\infty} (y_n(t) + (A_1' * y_n)(t) + (A_2 * y_n')(t)) = 0. \tag{3.5}$$

Let $\{t_m\}_{m=1}^\infty$ be an arbitrary sequence of positive numbers tending to $+\infty$. Then it follows from (2.5), (2.6), (3.4) and [**5**, Lemma 3.2] (applied to the function x), that there exists a subsequence, also denoted by $\{t_m\}_{m=1}^\infty$, and a Lipschitz continuous, bounded function w with a uniformly continuous derivative such that

$$y_n(t + t_m) \to w(t), \qquad y_n'(t + t_m) \to w'(t) \text{ as } m \to \infty$$
$$\text{uniformly on compact subsets of } \mathbf{R}. \tag{3.6}$$

Now we are going to show that w must be a constant. A combination of (3.5) and (3.6) yields

$$w + A_1' * w + A_2 * w' \equiv 0. \tag{3.7}$$

(Here "$*$" denotes convolution on the whole real line; the functions defined on $\mathbf{R}^+$ are extended as 0 on $(-\infty, 0)$). Let $u_0 \in \mathbf{R}$, $u_0 \neq 0$, be arbitrary and let p be a continuously differentiable function on $\mathbf{R}$ so that p and its derivative are integrable and $p\hat{}(u_0) \neq 0$. Taking the convolution of both sides of (3.7) with p and performing

an integration by parts one gets

$$(p + p * A_1' + p' * A_2) * w \equiv 0.$$

From this equation and the fact that the Fourier transform of $p + p * A_1' + p' * A_2$ does not vanish at u_0, since $p\hat{}(u_0) \neq 0$ and $A\hat{}(u_0) \neq 0$, we deduce that u_0 is not contained in the spectrum of w (i.e. the support of the distribution Fourier transform of w), see [**1**, p. 232]. Since $u_0 \neq 0$ was arbitrary it follows that the spectrum of w is contained in $\{0\}$; hence w is a constant as it is bounded.

Because the sequence $\{t_m\}_{m=1}^{\infty}$ was arbitrary and w is a constant, it follows from (2.13), (3.2) and (3.6) that $\lim_{t\to\infty}(A_2 * y_n)(t) = 0$. Thus we conclude that if (2.5), (2.6) and (3.3) imply (2.8), then $\lim_{t\to\infty} y_n(t) = 0$ for all n so that (2.5)–(2.7) also imply (2.8). The proof of the fact that if (2.5)–(2.7) imply (2.8) then (2.5), (2.6) and (3.3) imply (2.8) is almost exactly the same since it follows from (2.12) and (3.1) that $A_1\hat{}(u) \neq 0$, $u \in \mathbf{R}$.

Let us now assume that (2.14), or equivalently (2.15), holds. By Theorem 1 and the prior results it is sufficient to show that (2.3) holds with A replaced by A_1 and e.g. $c = 1$. Let $A_1'(t) = -a(t)$. Then

$$\begin{aligned} A_1\hat{}(u)(1 + A_1\hat{}(u))^{-1} &= \left((A_1\hat{}(u))^{-1} + 1\right)^{-1} \\ &= (iu + 1)^{-1}(1 - a\hat{}(u))\left(1 - (iu + 1)^{-1}a\hat{}(u)\right)^{-1} \end{aligned}$$

and a standard argument, using the facts that (2.15) holds, $a \in L^1(\mathbf{R}^+; \mathbf{R})$ and $1 + A_1\hat{}(u) \neq 0$ shows that $A_1\hat{}(u)(1 + A_1\hat{}(u))^{-1}$ is the Fourier transform of a function in $L^1(\mathbf{R}; \mathbf{C})$, see [**6**, pp. 60–63]. The proof of (2.3) will be completed once we can show that this integrable function is in fact supported on $\mathbf{R}^+$ and hence we must show that $\varphi(z) \overset{\text{def}}{=} A_1\hat{}(z)(1 + A_1\hat{}(z))^{-1}$ is continuous and bounded in $\operatorname{Im} z \leqslant 0$, cf. [**6**, pp. 60–63]. Since A_1 is of bounded variation and $1 + A_1\hat{}(z) \neq 0$ (because $A_1\hat{}(0) = +\infty$, $1 + A_1\hat{}(z) = 1 + (1 + A_1'\hat{}(z))/(iz)$ and $|A_1'\hat{}(z)| < 1$ if $\operatorname{Im} z \leqslant 0$ and $z \neq 0$ by (3.1) and (2.12)), it remains to check the continuity at $z = 0$. Clearly $\varphi(z)$ is analytic and nonzero (by (2.12) and (3.1)) in $\operatorname{Im} z < 0$ and as A_1 is of bounded variation one sees that $\varphi(z)$ belongs to the Nevanlinna classes in the lower halfspace, so that $\log|\varphi(z)|$ is a harmonic function that can be written as the Poisson integral of a measure (see [**7**, Chapter 17]). But now since $\log|\varphi(z)|$ is continuous in $\operatorname{Im} z \leqslant 0$, $z \neq 0$, at all points in a neighbourhood of $z = 0$ and also (by (2.15)) along the real axis it follows that $\log|\varphi(z)|$ must be continuous at $z = 0$ too. But this implies that $\varphi(z)$ is bounded at $z = 0$; hence $\varphi(z)$ belongs to the space H^∞ and is therefore the Poisson integral of its boundary values. Since $\varphi(z)$ is continuous in $\operatorname{Im} z \leqslant 0$, $z \neq 0$, and along the real axis, it follows that $\varphi(z)$ is continuous at $z = 0$ too. But this is what we wanted to show and the proof of the sufficiency of (2.14) is completed.

Next we assume that (2.14) and thus also (2.15) do not hold. We deduce that there cannot exist a function $B \in L^1(\mathbf{R}^+; \mathbf{R})$ such that

$$B\hat{}(z) = 1 - ((iz + 1)A_1\hat{}(z))^{-1}, \qquad \operatorname{Im} z \leqslant 0. \tag{3.8}$$

We have however, recalling that $A_1'(t) = -a(t)$,

$$B\hat{}(z) = (iz + 1)^{-1} + \big((iz + 1)^{-1} - 1\big)a\hat{}(z)(1 - a\hat{}(z))^{-1}, \qquad \operatorname{Im} z < 0, \quad (3.9)$$

and we conclude that there exists a continuous real function B so that $B\hat{}(z)$ is its Fourier transform. Moreover, since $a(t) \geq 0$, $\int_0^\infty a(t)\,dt = 1$ and $\int_0^\infty ta(t)\,dt = \int_0^\infty A_1(t)\,dt = +\infty$ it follows from (3.9) and the renewal theorem that

$$\int_t^{t+\tau} B(s)\,ds \to 0 \quad \text{as } t \to \infty \text{ for all } \tau > 0. \quad (3.10)$$

We are going to construct a bounded function g such that $\lim_{t\to\infty} g(t) = 0$ and $B * g$ is bounded but does not converge to 0. Then we take

$$x(t) = g(t) - (B * g)(t), \qquad t \in \mathbf{R}^+, \quad (3.11)$$

and note that by (3.8)

$$(A_1 * x)(t) = \int_0^t e^{-(t-s)} g(s)\,ds \to 0 \quad \text{as } t \to \infty. \quad (3.12)$$

For every integer $n \geq 1$ we define the function f_n by

$$f_n(t) = \operatorname{sign}(B(n - t)), \quad t \in [0, n], \qquad f_n(t) = 0, \quad t \notin [0, n].$$

Let

$$c_n = \sup_{t\in\mathbf{R}^+} |(B * f_n)(t)|, \qquad n = 1, 2, \ldots. \quad (3.13)$$

Since $B \notin L^1(\mathbf{R}^+; \mathbf{R})$ it follows from the definition of c_n that

$$\lim_{n\to\infty} c_n = +\infty, \quad (3.14)$$

but of course $c_n < \infty$ for all n. Now we choose a sequence $\{t_n\}_{n=1}^\infty$ of positive numbers such that $t_n - t_{n-1} \geq 2n$ and if

$$g(t) = \sum_{n=1}^\infty c_n^{-1} f_n(t - t_n), \qquad t \in \mathbf{R}^+, \quad (3.15)$$

then

$$\sup_{t\in\mathbf{R}^+} |(B * g)(t)| < 2, \qquad \limsup_{t\to\infty} |(B * g)(t)| \geq 2^{-1}. \quad (3.16)$$

This is possible to do because $\lim_{t\to\infty}(B * f_n)(t) = 0$ for all n (by (3.10)), and (3.13) and (3.15) hold (the same idea is used in [9]).

It follows from (3.14) that $\lim_{t\to\infty} g(t) = 0$ and therefore we conclude that if x is defined by (3.11) then (2.5) and (3.3) hold, see (3.12), but (2.8) does not hold, see (3.16). As we proved above that (2.7) could be replaced by (3.3) it remains to show that x satisfies (2.6).

To do this we use (3.9) to write $B(t) = B_1(t) + B_2(t)$, $t \in \mathbf{R}^+$, where

$$B_1\hat{}(z) = (iz + 1)^{-1}\big(1 - a\hat{}(z)(2 - a\hat{}(z))^{-1}\big)$$
$$+ \big((iz + 1)^{-1} - 1\big) \times 2a\hat{}(z)(2 - a\hat{}(z))^{-1},$$
$$B_2\hat{}(z) = a\hat{}(z)(2 - a\hat{}(z))^{-1} B\hat{}(z), \qquad \operatorname{Im} z < 0.$$

Since $\|a\|_{L^1(\mathbf{R}^+;\mathbf{R})} = 1$ (by (2.12) and (3.1)), we have $B_1 \in L^1(\mathbf{R}^+; \mathbf{R})$; hence $\lim_{t\to\infty}(B_1 * g)(t) = 0$. By the same argument we see that $a\hat{}(z)(2 - a\hat{}(z))^{-1}$ is the

Fourier transform of a function in $L^1(\mathbf{R}^+; \mathbf{R})$ and as $B * g$ is bounded it follows that $B_2 * g$ is uniformly continuous. If we combine these results with the fact that $\lim_{t\to\infty} g(t) = 0$ and the definition (3.11), then we see that (2.6) holds and the proof is completed.

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