EXTENSIONS OF PURE POSITIVE FUNCTIONALS
ON BANACH *-ALGEBRAS

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ABSTRACT. A known extension theorem for pure states on a Banach *-algebra with
isometric involution is shown to hold for the wider class of Banach *-algebras with
arbitrary, possibly discontinuous, involutions.

Let \( A \) be a Banach *-algebra with isometric involution and bounded approxi-
mate identity \( \{ e_a \} \), and \( B \) a closed *-subalgebra of \( A \) containing \( \{ e_a \} \). In [3] G.
Maltese proved that if \( f \) is a pure state on \( B \), then \( f \) admits a pure state extension to
\( A \) if and only if \( f \) admits a positive linear extension to \( A \). Our purpose here is to
extend this result to Banach *-algebras with arbitrary, possibly discontinuous,
involutions.

For basic definitions and results from the theory of Banach *-algebras and their
representations see [1], [2], or [4].

The following lemma handles the case when the algebra \( A \) contains an identity.

**Lemma 1.** Let \( A \) be a unital Banach *-algebra, \( B \) a closed *-subalgebra of \( A \)
containing the identity \( e \), and suppose that \( f \) is a pure positive linear functional on \( B \).
Then \( f \) can be extended to a pure positive linear functional on \( A \) if and only if \( f \) has a
positive linear extension to \( A \).

**Proof.** We may assume without loss of generality that \( f(e) = 1 \). Indeed, if \( \lambda > 0 \),
then \( \lambda f \) is pure and positive if \( f \) is pure and positive. Our proof will be given in two
steps:
I. \( A \) has continuous involution;
II. \( A \) has arbitrary involution.

**Proof of I.** Let \( P_A \) denote the set of positive functionals \( g \) on \( A \) satisfying
\( g(e) = 1 \). Define \( P_B \) similarly. It is well known that a functional in \( P_A \) (or \( P_B \)) is
pure (pure on \( B \)) if and only if it is an extreme point of \( P_A \) (\( P_B \)). Suppose, now, that
\( f \) has a positive linear extension to \( A \), and set \( X = \{ g \in P_A : g|_B = f \} \); i.e., \( X \) is the
set of all positive extensions of \( f \). Then \( X \) is nonempty by assumption, and it is
clearly convex. We show that \( X \) is compact in the relative weak *-topology. By the
Banach-Alaoglu theorem it suffices to show that \( X \) is weak *-closed and norm
bounded. Suppose that \( \{ g_a \} \) is a net in \( X \) and that \( g_a \to g \). Then by the definition
of the weak *-topology, \( g_a(x) \to g(x) \) for every \( x \in A \); thus, if \( x \in B \), then
\( g(x) = \lim_{a} g_a(x) = \lim_{a} f(x) = f(x) \) which implies \( g \in X \). Therefore, \( X \) is weak \(^*\)-closed. Now let \( g \in X \) be arbitrary; since \( x \to x^* \) is continuous there exists \( k > 0 \) such that \( \|x^*\| < k\|x\| \) for all \( x \) in \( A \). Then, by [4, pp. 214, 219],
\[
|g(x)|^2 < g(e)g(x^*x) < g(e)\nu(x^*x) < g(e)\|x^*x\| < g(e)k\|x\|^2 = k\|x\|^2,
\]
where \( \nu(\cdot) \) denotes the spectral radius. Hence \( \|g\| < \sqrt{k} \) and \( X \) is norm bounded.

The Krein-Milman theorem now implies that \( X \) has extreme points. We denote the set of extreme points of \( X \), \( P_A \), and \( P_B \) by \( E(X) \), \( E(P_A) \), and \( E(P_B) \) respectively. Verification of the equality \( E(X) = X \cap E(P_A) \) will complete the proof of Part I. Our proof follows that given in [3, p. 503].

It is clear that \( X \cap E(P_A) \subseteq E(X) \). Let \( g \in E(X) \) and suppose \( g = \frac{1}{2}(\phi + \psi) \), where \( \phi, \psi \in P_A \). Then, taking restrictions, we obtain \( f = \frac{1}{2}(\phi|_B + \psi|_B) \). But \( \phi|_B \) and \( \psi|_B \) are in \( P_B \), and since \( f \in E(P_B) \), it follows that \( f = \phi|_B = \psi|_B \) which implies that \( \phi \) and \( \psi \) are in \( X \). Since \( g \) is an extreme point of \( X \) we have \( g = \phi = \psi \) which means that \( g \in E(P_A) \). Hence \( E(X) \subseteq X \cap E(P_A) \).

**Proof of II.** We now allow the involution to be arbitrary. If \( J \) denotes the Jacobson radical of \( A \), then \( A/J \) is a semisimple Banach \(^*\)-algebra which, by Johnson's uniqueness of the norm theorem [1, p. 130], has continuous involution. Hence the closure \((B + J)/J^-\) in \( A/J \) of the \(^*\)-subalgebra \((B + J)/J\) is a Banach \(^*\)-subalgebra of \( A/J \) containing the identity \( e + J \).

Let \( f' \) be the positive extension of \( f \) to \( A \), and define a function \( \tilde{f}' : A/J \to \mathbb{C} \) by \( \tilde{f}'(x + J) = f'(x) \). We note that \( \tilde{f}' \) is well defined since \( f' \) is representable [4, p. 216] and \( J \) is contained in the reducing ideal. Furthermore, \( \tilde{f}' \) is linear and positive and is therefore continuous since \( A/J \) has an identity. Moreover, if \( b \in B \), then \( \tilde{f}'(b + J) = f'(b) = f(b) \). Let
\[
\tilde{f} = \tilde{f}'|_{((B + J)/J)^-}.
\]
Then \( \tilde{f} \) is a continuous positive linear functional on \((B + J)/J^-\) and \( \tilde{f}(b + J) = f(b) \) for every \( b \in B \). We assert that \( \tilde{f} \) is pure. Indeed, let \( \tilde{g} \) be an arbitrary positive functional on \((B + J)/J^-\) satisfying \( \tilde{g} \leq \tilde{f} \). Then \( \tilde{g}(b*b + J) \leq \tilde{f}(b*b + J) = f(b*b + J) \) for every \( b \in B \). Define a positive functional \( g \) on \( B \) by \( g(b) = \tilde{g}(b + J) \). Clearly \( g \leq f \), and since \( f \) is pure, it follows that \( g = \lambda f \), where \( 0 < \lambda < 1 \). Hence, \( \tilde{g} = \lambda \tilde{f} \) on \((B + J)/J\). But \( \tilde{g} \) and \( \tilde{f} \) are both continuous, and thus it follows that \( \tilde{g} = \lambda \tilde{f} \) on all of \((B + J)/J^-\); therefore, \( \tilde{f} \) is pure.

By part I, \( \tilde{f} \) has a pure positive extension to \( A/J \) which we denote by \( h \). Define \( h' : A \to \mathbb{C} \) by \( h'(x) = h(x + J) \). Then \( h' \) is a positive functional on \( A \) and if \( b \in B \), then \( h'(b) = h(b + J) = \tilde{f}(b + J) = f(b) \). It remains only to show that \( h' \) is pure. Let \( g' \) be a positive functional on \( A \) satisfying \( g' < h' \). Then \( g'(x^*x) < h'(x^*x) = h(x^*x + J) \). Define a functional \( g \) on \( A/J \) by \( g(x + J) = g'(x) \); \( g \) is well defined since \( g' \) is representable. Clearly \( g \) is positive and \( g < h \); but \( h \) is pure, so \( g = \lambda h \) which implies \( g' = \lambda h' \). Hence \( h' \) is pure and the proof is complete.

The next lemma is well known from Banach \(^*\)-algebras with isometric involution (see [2, 2.2.10, p. 34]). We give a simple proof for the case of an arbitrary involution. In what follows we assume that all bounded approximate identities are bounded by one.
Lemma 2. Let $A$ be a Banach *-algebra with bounded approximate identity $(e_a)$, a nondegenerate *-representation of $A$ on a Hilbert space $H$, and let $I$ denote the identity operator on $H$. Then $\lim_a \pi(e_a) = I$, where the limit is in the strong operator topology.

Proof. For each $x \in A$ we have $\|\pi(e_a) - \pi(x)\| < \|\pi\| \cdot \|e_a x - x\| \to 0$. Hence $\|\pi(e_a)\pi(x) - \pi(x)\| \to 0$ for every $x \in A$ and every $\xi \in H$. Since $\pi$ is nondegenerate, the set $\pi(A)H$ is dense in $H$. Now let $\eta \in H$ be arbitrary, $\varepsilon > 0$, and set $M = \max(\|\pi\|, 1)$. Then there exists $\xi \in H$ and $x \in A$ such that $\|\pi(x)\xi - \eta\| < \varepsilon/3M$ and there exists $a_0$ such that $\alpha > a_0$ implies $\|\pi(e_a)\pi(x)\xi - \pi(x)\xi\| < \varepsilon/3$.

Then
\[
\|\pi(e_a)\eta - \eta\| < \|\pi(e_a)\eta - \pi(e_a)\pi(x)\xi\| + \|\pi(e_a)\pi(x)\xi - \pi(x)\xi\| + \|\pi(x)\xi - \eta\| < \|\pi\| \cdot \|e_a\| \cdot \|\eta - \pi(x)\xi\| + \varepsilon/3 + \varepsilon/3M < \varepsilon.
\]
completing the proof.

Theorem 3. Let $A$ be a Banach *-algebra with bounded approximate identity $(e_a)$ and suppose $B$ is a closed *-subalgebra of $A$ containing $(e_a)$. Let $f$ be a pure positive linear functional on $B$ admitting a positive linear extension $f'$ to $A$. Then $f$ has a pure positive linear extension to $A$.

Proof. Since $f$ and $f'$ are representable, we can write $f(b) = (\pi(b)\xi|\xi)$ and $f'(x) = (\pi'(x)\xi'|\xi')$ for all $b \in B$, $x \in A$, and suitable vectors $\xi$ and $\xi'$ in the respective spaces of $\pi$ and $\pi'$. Then, by Lemma 2, $\|\xi\| = \|\xi\| = \lim_a \pi(e_a) = \lim_a \pi'(e_a) = \|\xi\| = \|\xi\|$. Let $A_e$ and $B_e$ denote the Banach *-algebras obtained from $A$ and $B$ respectively by adjoining identities. Define *-representations $\pi'$ and $\pi$ of $A_e$ and $B_e$ respectively by $\pi'(x, \lambda) = \pi'(x) + \lambda I$ and $\pi((b, \lambda)) = \pi(b) + \lambda I$, where $I$ denotes the identity operator. Let $\tilde{f}'(x, \lambda) = (\pi'(x, \lambda)\xi'|\xi')$ and $\tilde{f}(b, \lambda) = (\pi(b, \lambda)\xi|\xi)$. Then $\tilde{f}'$ and $\tilde{f}$ are positive functionals on $A_e$ and $B_e$ respectively and
\[
\tilde{f}'(b, \lambda) = (\pi'(b, \lambda)\xi'|\xi') = (\pi(b)\xi|\xi') + \lambda(\xi'|\xi') = (\pi(b)\xi|\xi') + \lambda(\xi'|\xi')
\]
for every $(b, \lambda) \in B_e$. Now $f$ pure implies that $\pi$ is irreducible [2, 2.5.4, p. 43]; hence $\pi'$ is irreducible and thus $\tilde{f}'$ is pure. By Lemma 1, $\tilde{f}$ has a pure positive extension, say $g$, to $A_e$. Hence there exists an irreducible *-representation $\pi_g$ and a cyclic vector $\xi_c$ such that $g((x, \lambda)) = (\pi_g((x, \lambda))\xi_c|\xi_c)$. So $\pi_{g|A}$ is also irreducible, and therefore the functional $g_A$ defined on $A$ by $g_A(x) = (\pi_{g|A}(x)\xi_c|\xi_c)$ is a pure positive functional on $A$. Moreover, $g_A(b) = g((b, 0)) = \tilde{f}'(b, 0) = (\pi'(b, 0)\xi|\xi) = \pi(b)\xi|\xi = f(b)$. 

REFERENCES


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