ELEMENTARY PROOF OF THE RUDIN-CARLESON
AND THE F. AND M. RIESZ THEOREMS

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Abstract. A very elementary proof is given of the theorem that on a set of
measure zero on $T$, any continuous function is equal to a continuous function of
analytic type. The same elementary method proves that a measure of analytic type
is absolutely continuous.

A complex Borel measure $\mu$ on $T$, in particular an $f \in L^1(T)$, is said to be of
analytic type if

$$a_n = (2\pi)^{-1} \int_T e^{-int} d\mu(t) = 0, \quad n = -1, -2, \ldots.$$ 

The theorems mentioned in the title are:

RUDIN-CARLESON Theorem. Let $F$ be a closed subset of $T$ of Lebesgue measure
zero. If $\varphi$ is a continuous function on $F$, then there is a continuous function $f$, of
analytic type, such that

$$f(t) = \varphi(t), \quad t \in F,$$

$$\sup_{t \in F} |f(t)| \leq M \sup_{t \in F} |\varphi(t)| \quad (\ast)$$

where $M$ is a constant. (Rudin proves that $M = 1$. See [8] and [1].)

The First F. and M. Riesz Theorem. If the function $f$ in $L^1(T)$ is of analytic type
and if $f$ vanishes on a set $S^*$ of positive measure, then $f = 0$.

The Second F. and M. Riesz Theorem. If a complex Borel measure $\mu$ on $T$ is of
analytic type, then $\mu$ is absolutely continuous (with respect to Lebesgue measure). See
[7].

The proofs of these theorems most often use boundary values of functions
analytic in the unit disc and the theory of $H^p$-spaces. For the Second F. and M.
Riesz Theorem, for example, see three variants in [3], [5] and [9]; other proofs of
that theorem use Hilbert-space theory: see e.g. [2] and [4]; a direct short proof is
given in [6].

The aim of the present paper is to present a method which gives an elementary
proof of all the above theorems.

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**Lemma.** Let $F$ be a closed subset of $T$ of measure zero and $\varphi$ a continuous function on $F$. Given $\varepsilon > 0$ and an open set $G \supseteq F$ there is a continuous function $g$ of analytic type such that

\[
\sup_{t \in F} |g(t) - \varphi(t)| < \varepsilon \sup_{t \in F} |\varphi(t)|, \\
|g(t)| < \varepsilon, \quad t \notin G, \\
\sup_{t \in T} |g(t)| < 3 \sup_{t \in F} |\varphi(t)|.
\]

**Proof.** Without loss of generality we may assume that $\sup_{t \in F} |\varphi(t)| = 1$ and also that $\varphi$ is a trigonometric polynomial

\[
\varphi(t) = \sum_{|k| < m} a_k e^{ikt}
\]

such that

\[
|\varphi(t)| < \varepsilon/3, \quad t \notin G.
\]

Let $e^{-A} = \varepsilon$ and let $h$ be a continuous function on $T$, lying between $-2A$ and $2\varepsilon$, such that

\[
|h(t) + 2A| < \varepsilon, \quad t \notin F.
\]

Since $m(F) = 0$ we may take $\|h\|_1$ arbitrarily small and hence we may suppose $\tilde{h}(k) = 0$, $|k| < m$. Take a Fejér sum $p$ of $h$ such that $|p(t) + 2A| < \varepsilon$, $t \in F$. We write

\[
p(t) = \sum_{k < -m} \beta_k e^{ikt} + \sum_{k > m} \beta_k e^{ikt} = p^-(t) + p^+(t)
\]

where

\[
p^+(t) = \sum_{k > m} \beta_k e^{ikt}.
\]

We have $\Re(p^+) = p/2 < \varepsilon$. Put now

\[
g(t) = \varphi(t)[1 - e^{p^+(t)}].
\]

The expansion of $[1 - e^{p^+(t)}]$ is of the form $\sum_{k > m} \gamma_k e^{ikt}$. The function $g$ is therefore continuous of analytic type. We have

\[
|g(t) - \varphi(t)| = |\varphi(t)||1 - e^{p^+(t)}| < e^{p/2} < e^{2A + \varepsilon} < 2\varepsilon \quad (t \in F).
\]

Moreover

\[
|g(t)| < |\varphi(t)| |1 - e^{p^+(t)}| < 1 + \varepsilon < 3 \quad (t \in T).
\]

\[
|g(t)| < (\varepsilon/3)3 = \varepsilon \quad (t \notin G).
\]

The Lemma is now proved.

**Proof of the Rudin-Carleson Theorem.** $\varepsilon < \frac{1}{4}$ being fixed, denote by $\gamma(\varphi)$ any continuous function of analytic type associated to $\varphi$ by the Lemma. Starting with $\varphi_0 = \varphi$ we put $\varphi_{m+1} = \varphi_m - \gamma(\varphi_m)$. We have

\[
\sup_{F} |\varphi_{m+1}| < \varepsilon \sup_{F} |\varphi_{m}| < \cdots < \varepsilon^{m+1} \sup_{F} |\varphi_0|, \\
\sup_{F} |\gamma(\varphi_m)| < 3 \sup_{F} |\varphi_m| < 3\varepsilon^m \sup_{F} |\varphi_0|.
\]
The series $\sum_{m=0}^{\infty} \gamma(q_m)$ is therefore uniformly convergent on $T$; its sum $f$ is of analytic type and satisfies the relation $f(t) = \varphi(t)$ ($t \in F$). Moreover

$$\sup_T |f(t)| < 3(1 - \epsilon)^{-1} \sup_F |\varphi_0| < 4 \sup_F |\varphi|.$$  

The theorem is now proved.

**Remark.** The factor $M$ in the estimate (*) can easily be reduced to $1 + \epsilon$. In fact, given an open set $G \supset F$ and using (**) we can manage to have

$$|f(t)| < \epsilon \quad (t \not \in G).$$

By the continuity of $f$, there is an open set $G' \supset G$ such that $G' \subset G$ and $|f(t)| < 1 + \epsilon$ ($t \in G'$). Thus we can have

$$|f(t)| > 1 + \epsilon \quad \text{only if } t \in G \setminus G'.$$

Starting with $G'$ we get $f'$ coinciding with $\varphi$ on $F$, bounded by 4 where $|f'(t)| > 1 + \epsilon$ only if $t \in G' \setminus G''$ for an appropriate $G'' \supset G'$. Observing that the sets $G \setminus G', G' \setminus G'', G'' \setminus G''', \ldots$ are disjoint and taking an arithmetic mean we get a function bounded everywhere by $1 + 2\epsilon$.

**Proof of the First F. and M. Riesz Theorem.** It is sufficient to prove that

$$a_0 = (2\pi)^{-1} \int_T f(t) \, dt = 0$$

for, applying the same process to the function $e^{-\varphi f(t)}$, we deduce $a_1 = 0$, and next $a_2 = 0, \ldots$ and finally $f = 0$. We shall follow the same pattern of proof as for the Rudin-Carleson Theorem.

Denote by $S$ the set \{ $t \in T$: $f(t) \neq 0$ \}. Given $\epsilon > 0$ let $e^{-A} = \epsilon$ and let $h$ be a bounded real function equal to $-2A$ on $S$ and such that $\hat{h}(0) = 0$. There are such functions since $m(S^*) > 0$. Let $p_n$ be the sequence of Fejér polynomials of $h$. We write as before

$$p_n(t) = \sum_{k<0} \beta_k e^{ikt} + \sum_{k>0} \beta_k e^{ikt} = p_n^-(t) + p_n^+(t)$$

where

$$p_n^+(t) = \sum_{k>0} \beta_k e^{ikt}.$$  

Then, boundedly,

$$\Re(p_n^+(t)) = \frac{1}{2} p_n(t) \to \frac{1}{2} h(t) = -A \quad \text{a.e. on } S.$$  

Put now

$$g_n(t) = f(t)[1 - e^{2\pi A(t)}].$$  

The expansion of $g_n$ is of the form $\sum_{k>0} \gamma_k e^{ikt}$ and therefore $\int g_n \, dt = 0$. Hence

$$|2\pi a_0| = \left| \int f \right| = \left| \int (f - g_n) \right| < \left| \int e^{2\pi A} \right|$$

$$< \int_S |f| e^{\pi A} \to e^{-A} \int |f| = \epsilon \|f\|_1.$$  

Since $\epsilon$ is arbitrary we have $a_0 = 0$ and the theorem is proved.
Proof of the Second F. and M. Riesz Theorem. We may assume $a_0 = 0$. Let $F$ be a closed set of measure zero. Choose a decreasing sequence of open sets $G_n \supset F$ such that $\cap G_n = F$, and by the Lemma a sequence of functions $g_n$ of analytic type, such that

$$
|1 - g_n(t)| < 1/n, \quad t \in F, \\
|g_n| < 3; \quad |g_n(t)| < 1/n \quad \text{for } t \not\in G_n.
$$

Then, boundedly, $g_n \to \chi_F$ (characteristic function of $F$). Hence $0 = \int g_n \, d\mu \to \int \chi_F \, d\mu = \mu(F)$. This proves that $\mu$ is absolutely continuous.

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References

7. F. Riesz and M. Riesz, Über die Randwerte einer analytischen Funktion, 4e Congrès des Mathématiciens Scandinaves (Stockholm, 1916), pp. 27–44.

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