DEFICIENT VALUES OF ENTIRE FUNCTIONS AND THEIR DERIVATIVES

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Abstract. Let \( f(z) \) be entire and of finite order, \( f^{(n)} \) be the \( n \)th derivative, and \( \Delta_n(f) = \sum \delta(a, f^{(n)}) \), the sum of all deficient values of \( f^{(n)} \). The authors show that \( \Delta_n(f) \) can be strictly increasing.

Let \( f(z) \) be entire of order \( \rho < \infty \), and for \( 0 < j < \infty \) let

\[
\Delta_j(f) = \sum_{|a| < \infty} \delta(a, f^{(j)}),
\]

where \( f^{(0)} = f \) and \( f^{(j)} \) is the \( j \)th derivative. Using the relation \([4, \text{p. 104}] \sum \delta(a, f) < \delta(0, f')\), it is clear that \( \Delta_j(f) \) is nondecreasing in \( j \) while \( \Delta_j(f) < 1 \) for all \( j \). Professor W. H. J. Fuchs \([7, \text{p. 167}]\) recently asked if it is possible that \( \Delta_j(f) \) be strictly increasing. In this paper we give an affirmative answer. More precisely, we have the stronger

**Theorem.** Let \( c_{jk} \) \((j = 0, 1, 2, \ldots; k = 1, 2, \ldots, K_j; 1 < K_j < \infty)\) be finite complex numbers, with \( c_{jk} \neq c_{jk'} \) \((k \neq k')\). Given \( \frac{1}{2} < \rho < \infty \), and an increasing sequence \( \{n_j\} \) of integers, there exists an entire function \( f(z) \) of order \( \rho \), mean type, such that

\[
\delta(c_{jk}, f^{(n_j)}) > 0
\]

for all \( j \) and \( k \).

Recently, two of us \([8]\) proved that if \( \Delta = \lim \Delta_j(f) = 1 \), then \( \Delta_j(f) \equiv 1 \) for \( j > \text{max}(f) \). In the example here \( \Delta \) is considerably less than 1.

Our proof is based on N. Arakelyan’s method \([2]\) which produces entire functions of finite order having an infinite set of deficient values. Here, we have a set of deficient functions rather than numbers, but Arakelyan’s method is sufficiently flexible to adapt to this situation. The restriction \( \rho > \frac{1}{2} \) is essential, since if \( \rho < \frac{1}{2} \), then \( \Delta_j(f) \equiv 0 \) for all \( j \).
1. Preliminary propositions.

**Proposition 1 (Mergelyan [6, p. 125]).** Let \( \mathcal{L} : z = z(t), 0 < t < 1, \) be a simple rectifiable curve of length \( L, z(0) = a, z(1) = b. \) If \( d > 0 \) and \( 0 < \varepsilon < 1, \) there exists a polynomial \( P(z) \) such that
\[
\left| \frac{1}{z-a} - P\left( \frac{1}{z-b} \right) \right| < \varepsilon \tag{1.1}
\]
holds except in a \( d \)-neighborhood of \( \mathcal{L} \) and also\(^2\)
\[
\left| P\left( \frac{1}{z-b} \right) \right| < \exp\left( \left( 1 + \log\left( 1 + \frac{1}{ed} \right) \right) e^{A(L/d) + A} \right) (|z - b| > d). \tag{1.2}
\]

**Proposition 2 (Mergelyan [5, p. 61]).** Let \( f(z) \) be analytic in the sector \( |\arg z| < \alpha/2, \) let the number \( p \) satisfy the condition \( 0 < p < \pi/\alpha, \) and \( \varepsilon > 0, \eta > 0 \) be any numbers. Then there exists an entire function \( G(z) \) with
\[
|f(z) - G(z)| < \varepsilon \exp(-|z|\eta) \tag{1.3}
\]
in the sector \( |\arg z| < \alpha/2 - \eta \) and
\[
\log|G(z)| < (1 + r)^{\eta/(2\pi - \alpha)} \left( K + k \max_{0 < r < kr+1} \frac{t^\rho + \log^+ M(t, f)}{(1 + t)^{\eta/(2\pi - \alpha)}} \right) \tag{1.4}
\]
in the whole plane; in (1.4) \( k \) is a constant depending on \( \eta, K \) depends on \( \varepsilon \) and \( \eta, \) and \( M(t, f) = \max|f(te^\theta)| (|\theta| < \alpha/2). \)

2. Proof of the theorem.

2.1. It is no loss of generality to assume \( n_j = j. \)

Choose \( 0 < \alpha < \min(\pi/\rho, 2\pi - \pi/\rho) \) and \( \gamma^j (j = 0, 1, 2, \ldots) \) such that
\[
0 < \gamma^0 < \gamma^1 < \cdots < \alpha/2,
\]
and then, for each \( j, \) choose
\[
\gamma^j < \gamma_1 < \gamma_2 < \cdots < \gamma_{K_j} < \gamma^{j+1}.
\]

Then we let
\[
\gamma_{j,-k} = -\gamma_{jk}, \quad (j = 0, 1, 2, \ldots; k = 1, 2, \ldots, K_j),
\]
\[
\alpha_{jk} = \min\left\{ \frac{1}{2}(\gamma_{jk+1} - \gamma_{jk}), \frac{1}{2}(\gamma_{jk} - \gamma_{jk-1}) \right\} = \alpha_{j,-k},
\]
where
\[
\gamma_{jk+1} = \gamma^{j+1}, \quad \gamma_0 = \gamma^j \quad (j = 0, 1, 2, \ldots).
\]

Put
\[
E_{jk} = \left\{ re^{i\theta} : 2^n < r < 2^{n+1}, |\theta - \gamma_{jk}| < (1/16)\alpha_{jk} \right\},
\]
\[
E^1_{jk} = \left\{ re^{i\theta} : (15/16)2^n < r < (17/16)2^{n+1}, |\theta - \gamma_{jk}| < (1/8)\alpha_{jk} \right\},
\]
\[
E^2_{jk} = \left\{ re^{i\theta} : (7/8)2^n < r < (9/8)2^{n+1}, |\theta - \gamma_{jk}| < (1/4)\alpha_{jk} \right\}.
\]

We will construct an entire function \( f(z) \) which satisfies
\[
|f(z)| < \exp\{A(|z|^\rho + 1) \} \tag{2.1}
\]
\(^2\)Here and henceforth \( A \) denotes a generic positive absolute constant.
for all $z$ and, for a positive sequence $\varepsilon_{jk}$ to be determined by (2.5),

$$|f(z) - (c_{jk}/j!)z^j| < 2 \exp\{-A\varepsilon_{jk}|z|^\rho\}$$

(2.2)

for

$$z \in E_{jkn}^1 \quad (n > n_{jk}, \text{n even}),$$

$$z \in E_{j-k,n}^1 \quad (n > n_{jk}, \text{n odd}).$$

(2.3)

The $n_{jk}$ are chosen precisely in (2.15) below. Let $H$ be the set of $(j, k, n)$ which appear in (2.3) and denote a typical element $(j, k, n)$ of $H$ by $h$. We see that $f(z)$ is our required function. In fact, when $z \in E_h$, a disk with center at $z$ and radius $10^{-2}a_{jk}2^n$ is contained completely in $E_{jkn}^1 = E_{h}^1$. According to Cauchy’s inequality and (2.2),

$$|(f(z) - (c_{jk}/j!)z^j)^{(j)}| < 2 \cdot 10^{2j!} \frac{\exp\{-A\varepsilon_{jk}|z| - (10)^{-2}a_{jk}2^n|^\rho\}}{(a_{jk}2^n)^j}$$

for

$$z \in E_h,$$

i.e.

$$\frac{1}{|f^{(j)}(z) - c_{jk}|} > \frac{A}{(2 \cdot 10^{2j!})} \frac{a_{jk}2^n \exp\{A\varepsilon_{jk}|z|^\rho\}}{(a_{jk}2^n)^j} \quad (z \in E_h),$$

where $h = (j, k, n) \in H$. On noting from (2.1) that

$$T(r, f^{(j)}) = m(r, f^{(j)}) < m(r, f) + m(r, f^{(j)}/f) < A(r^\rho + 1),$$

we obtain by integrating over $(|z| = r) \cap E_h$

$$\delta(c_{jk}, f^{(j)}) > A\alpha_{jk}\varepsilon_{jk} > 0 \quad (j = 0, 1, 2, \ldots; k = 1, 2, \ldots, K_j).$$

2.2. We now construct a function $Q(\xi, z)$. When $h = (j, k, n) \in H$, let $C_h$ be the arc of the circle $|z| = (9/8)2^{n+1}$ linking $\partial E_h^2$ to the point $z = -(9/8)2^{n+1}$ which does not meet the positive axis, and let $D_h$ be the $(2^{n-5}\alpha_{jk})$-neighborhood of $C_h \cup \partial E_h^2$. If $\xi$ is an arbitrary point of $\partial E_h^2$, we may connect $\xi$ to $z = -(9/8)2^{n+1}$ by a curve contained in $C_h \cup \partial E_h^2$. This curve has length less than $A \cdot 2^n$, so Proposition 1 produces a rational function $Q(\xi, z)$ with a unique pole at $-(9/8)2^{n+1}$ such that

$$|Q(\xi, z) - 1/(\xi - z)| < \eta_h \quad (\xi \in \partial E_h^2, z \notin D_h).$$

(2.4)

Thus let

$$\varepsilon_{jk} = \exp(-A/\alpha_{jk})$$

(2.5)

so that $\varepsilon_{jk} > 0$ and

$$\sum_{j,k} \varepsilon_{jk} = \sum_{j,k} \exp(-A/\alpha_{jk}) < \sum_{j,k} \alpha_{jk}/A < A.$$

Then we choose

$$\eta_h = \eta_{jkn} = \alpha_{jk}^{-1}12^{-1}(n+2n+2j) \exp\{-A\varepsilon_{jk}2^n\rho\},$$
and observe that
\[ \left\{ 1 + \log\left(1 + \frac{A}{\eta_h 2^n \alpha_{jk}}\right) \right\} \exp\left(\frac{A}{\alpha_{jk}}\right) \leq A 4^\nu e_{jk} 2^n \exp\left(\frac{A}{\alpha_{jk}}\right) \leq A 4^\nu e_{jk}^{1/2} \]

\[ (n > n_{jk}). \]

Further, recall the choice of \( \alpha \) from the beginning of §2.1. Then if \(|\arg z| < \alpha/2\), it is clear that \(|z - (-9/8)2^n| > A 2^n + 1\). With these choices of \( \varepsilon \) and \( \eta_h \) in (1.2) and (2.4) we have
\[
\left| Q(\zeta, z) - \frac{1}{\zeta - z} \right| < \alpha_{jk}^{-1} 2^{-(jn + 2n + 2j)} \exp\left\{-4^\nu e_{jk} \cdot 2^n \right\} \\
\left( \zeta \in \partial E_h^2, z \notin D_h \right), \quad (2.6)
\]
\[
|Q(\zeta, z)| < \exp\left(4^\nu e_{jk}^{1/2} 2^n \right) \quad \left( |\arg z| < \alpha/2, \zeta \in \partial E_h^2 \right). \quad (2.7)
\]

2.3. The next proposition is essentially in [1], [2, p. 96].

**Proposition 3.** Let \( 0 < \alpha < \min(\pi/\rho, 2\pi - \pi/\rho) \), \( \gamma_{jk} \) and \( \alpha_{jk} \) be as in §2.1, and set
\[
\psi(z) = \exp\left(-e_{jk} z^p\right) \quad \left( |\arg z - \gamma_{jk}| < \frac{\alpha}{4} \alpha_{jk} \right). \quad (2.8)
\]
Then there exists a function \( \omega(z) \) holomorphic in \(|\arg z| < \alpha/2\) such that
\[
|\omega(z)| < \exp(1 + |z|^p) \quad \left( |\arg z| < \alpha/2 \right) \quad (2.9)
\]
and for \( h = (j, k, n) \in H \)
\[
A < |\omega(z)/\psi(z)| < A \quad \left( z \in \bigcup \limits_{H} E_h^2 \right). \quad (2.10)
\]

**Proposition 4.** We can choose \( n_{jk} \) so that if
\[
g(z) = (C_j / j!) z^j \quad \left( z \in E_h^2, n > n_{jk} \right), \quad (2.11)
\]
there exists a function \( F(z) \) analytic in \(|\arg z| < \alpha/2\) such that
\[
|F(z)| < \exp A(|z|^p + 1) \quad \left( |\arg z| < \alpha/2 \right) \quad (2.12)
\]
and
\[
|F(z) - (C_{jk} / j!) z^j| < |\exp(-e_{jk} z^p)| \quad \left( z \in E_h^2, n > n_{jk} \right). \quad (2.13)
\]

**Proof.** Let \( \omega(z) \) be the function obtained in Proposition 3. Then if \( h = (jkn) \in H \), and
\[
g_h(z) = \frac{1}{2\pi i} \int \frac{g(\zeta)}{\omega(\zeta)} Q(\zeta, z) \, d\zeta,
\]
Proposition 3 shows that \( g_h \) is analytic in \(|\arg z| < \alpha/2\). Using (2.6), (2.8), (2.10) and (2.11), we can obtain
\[
\left| g_h(z) - \frac{1}{2\pi i} \int \frac{g(\zeta)}{\omega(\zeta)(\zeta - z)} \, d\zeta \right| < A \int_{\partial E_h^2} |\bar{c}_{jk}| |s|^{-1} \exp\left\{-4^\nu e_{jk} \cdot 2^n \right\} \frac{|d\zeta|}{2^{(n + 2n + 2j)}} \]
\[
< A |c_{jk}| \alpha_{jk}^{-1} \exp(-e_{jk} \cdot 2^n) \cdot 2^{-n} \quad \left( z \in D_h^1 \right). \quad (2.14)
\]
Choose \( n_jk \) so that
\[
\alpha_{jk}^{-1} \exp(-\varepsilon_{jk} \cdot 2^{n_p}) < \exp\left\{-\left(\varepsilon_{jk}/2\right) \cdot 2^{n_p}\right\} \quad (n > n_jk)
\] (2.15)
and that
\[
A \sum_H |c_{jk}| 2^{-n} < 1.
\] (2.16)

The integral in the left-hand side of (2.14) is given by
\[
\frac{1}{2\pi i} \int_{\partial E^2_k} \frac{g(\xi)}{\omega(\xi)(\xi - z)} \, d\xi = \begin{cases} 0 & \text{if } z \notin E^2_h, \\ g(z)/\omega(z) & \text{if } z \in E^2_h. \end{cases}
\] (2.17)

When \( z \in E^2_h \cup D^1_h \), we have \( 2^{n-1} \leq |z| < 2^{n+2} \), so that
\[
|g_h(z)| < A|c_{jk}|2^{(n+2)/2}\exp\left(A4^p\varepsilon_{jk}/2^{n_p}\right) \cdot \exp(\varepsilon_{jk}2^{n_p})2^n < |c_{jk}|2^{-n}\exp\left(A4^p\varepsilon_{jk}/2^{n_p}\right)
\] (2.18)
by (2.7) and Proposition 3.

Consequently, from (2.14), (2.16), (2.17) and (2.18),
\[
G(z) = \sum_H g_h(z)
\]
is absolutely convergent for every compact region in the angle \( |\arg z| < \alpha/2 \) and \( G(z) \) is analytic in this angle. Moreover, from (2.14), (2.17) and (2.18), we have
\[
|G(z)| < \sum_H |g_{jk}(z)| < \exp(A(1 + |z|^p))
\]
and
\[
\left| G(z) - \frac{c_{jk}}{j!} \frac{z^j}{\omega(z)} \right| < 1 \quad (z \in E^4_h).
\]
Thus if \( F(z) = G(z)\omega(z) \), then \( F(z) \) satisfies the conditions of Proposition 4.

2.4. In order to complete the proof of the theorem, we apply Proposition 2. There exists an entire function \( f(z) \) such that
\[
|f(z) - F(z)| < \exp(-|z|^p) \] (2.19)
in the angle \( |\arg z| < \alpha/2 - \eta \) and
\[
\log|f(z)| < (1 + r)^\pi/(2\pi - \alpha) \left\{ K + k \max_{0 < t < k r + 1} \frac{t^\rho + \log^+ M(t, F)}{(1 + t)^\pi/(2\pi - \alpha)} \right\}
\]
in the whole plane.

On noting (2.12) and
\[
\max_{0 < t < kr + 1} \frac{t^\rho}{(1 + t)^\pi/(2\pi - \alpha)} < Akr^\rho < \frac{r^\rho}{(1 + r)^\pi/(2\pi - \alpha)}
\]
we have (2.1). In every \( E^1_h \), we obtain from (2.13) and (2.19)
\[
|f(z) - (c_{jk}/j!)z^j| < e^{-r^\rho} + |F(z) - (c_{jk}/j!)z^j|
\]
\[
< e^{-r^\rho} + \exp(-A\varepsilon_{jk}r^\rho) < 2 \exp(-A\varepsilon_{jk}r^\rho).
\]
Thus (2.2) is satisfied and so \( f(z) \) is our desired function.
REFERENCES


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